

# SINGULARITY CATEGORIES OF DEFORMATIONS OF KLEINIAN SINGULARITIES

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**ABSTRACT.** Let  $G$  be a finite subgroup of  $\mathrm{SL}(2, \mathbb{k})$  and let  $R = \mathbb{k}[x, y]^G$  be the coordinate ring of the corresponding Kleinian singularity. In 1998, Crawley-Boevey and Holland defined deformations  $\mathcal{O}^\lambda$  of  $R$  parametrised by weights  $\lambda$ . In this paper, we determine the singularity categories  $\mathcal{D}_{\mathrm{sg}}(\mathcal{O}^\lambda)$  of these deformations, and show that they correspond to subgraphs of the Dynkin graph associated to  $R$ . This generalises known results on the structure of  $\mathcal{D}_{\mathrm{sg}}(R)$ .

The intersection theory of the minimal resolution of  $\mathrm{Spec} R$  is well understood via the geometric McKay correspondence. We show that the most singular  $\mathcal{O}^\lambda$  may be viewed as a noncommutative analogue of  $R$ , and in this case we show that  $\mathcal{O}^\lambda$  has a noncommutative resolution for which an analogue of the geometric McKay correspondence holds.

## 1. INTRODUCTION

**1.1. Background.** Throughout let  $\mathbb{k}$  be an algebraically closed field of characteristic zero. The Kleinian singularities  $\mathbb{k}^2/G$ , where  $G$  is a finite subgroup of  $\mathrm{SL}(2, \mathbb{k})$ , are ubiquitous in algebraic geometry, representation theory, and singularity theory. In this paper, we shall study the latter of these for a family of (generically noncommutative) algebras.

The notion of the singularity category of a ring  $R$  was introduced by Buchweitz in [Buc86] as a particular Verdier quotient of  $\mathcal{D}^b(\mathrm{mod}\text{-}R)$ . Specifically, writing  $\mathrm{Perf}(R)$  for the full subcategory of perfect complexes in  $\mathcal{D}^b(\mathrm{mod}\text{-}R)$ , Buchweitz defined the singularity category as the Verdier quotient category

$$\mathcal{D}_{\mathrm{sg}}(R) := \frac{\mathcal{D}^b(\mathrm{mod}\text{-}R)}{\mathrm{Perf}(R)}.$$

By construction, this category possesses the structure of a triangulated category.

From the above definition, it is not difficult to see that  $\mathcal{D}_{\mathrm{sg}}(R)$  is trivial precisely when  $R$  has finite global dimension. However, in general it is difficult to give an adequate description of the singularity category of an arbitrary Gorenstein ring of infinite global dimension. Recent work includes that of Chen [Che11, Che15] who has described the singularity category when  $R$  has radical square zero or when it is a quadratic monomial algebra, and that of Kalck [Kal15] who has provided a description when  $R$  is a so-called gentle algebra. In [AIR15] the authors provide a description of the singularity category of a Kleinian singularity which we recover, and they also determine the singularity categories of some related higher-dimensional analogues.

The main aim of this paper is to provide a concrete description of the singularity categories of certain deformations of the coordinate ring of a Kleinian singularity. In [CBH98], Crawley-Boevey and Holland defined a family of algebras  $\mathcal{O}^\lambda(\tilde{Q})$  depending on the data of an extended Dynkin quiver  $\tilde{Q}$  and a so-called weight for  $\tilde{Q}$ . Write  $Q$  for the Dynkin quiver obtained from  $\tilde{Q}$  by removing an extending vertex, and write  $R_Q$  for the coordinate ring of the corresponding Kleinian

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singularity. Then the algebras  $\mathcal{O}^\lambda(\tilde{Q})$  may be thought of as deformations of  $R_Q$  in the sense that there exists a filtration  $\mathcal{F}$  of  $\mathcal{O}^\lambda(\tilde{Q})$  satisfying  $\text{gr}_{\mathcal{F}} \mathcal{O}^\lambda(\tilde{Q}) = R_Q$ . These deformations are frequently noncommutative, a property which depends on the weight  $\lambda$ , and it is easy to determine when this is the case. If  $\mathcal{O}^\lambda(\tilde{Q})$  is commutative, a description of its singularity category follows from [IW14, Theorem 3.2], where quite geometric techniques are employed. Through a completely ring-theoretic approach, we determine  $\mathcal{D}_{\text{sg}}(\mathcal{O}^\lambda(\tilde{Q}))$  irrespective of whether the deformation is commutative or noncommutative.

Our other main result generalises the intersection theory of a minimal resolution of  $\text{Spec } R_Q$ , which is often referred to as the geometric McKay correspondence. It is well-known (see, for example, [LW12, §6.4]) that the exceptional fibre of the minimal resolution of the singular surface  $\text{Spec } R_Q$  is a union of  $n$  irreducible curves  $\gamma_i$ , where  $n$  is the number of vertices of  $Q$ . Moreover, each  $\gamma_i$  is isomorphic to  $\mathbb{P}^1$  and has self-intersection  $-2$ , and  $\gamma_i \cap \gamma_j$  is either empty or a point. In fact, the dual graph of the exceptional fibre is given by the underlying graph of  $Q$ . Let  $\Gamma$  be the  $n \times n$  matrix with entries

$$\Gamma_{ij} = \begin{cases} -2 & \text{if } i = j \\ 1 & \text{if } \gamma_i \text{ and } \gamma_j \text{ intersect} \\ 0 & \text{otherwise} \end{cases},$$

so that  $\Gamma_{ij}$  is equal to the intersection multiplicity  $\gamma_i \bullet \gamma_j$ . With an appropriate labelling of the curves  $\gamma_i$ , we have  $\Gamma = -C$  where  $C$  is the Cartan matrix corresponding to the Dynkin type of  $Q$ ; explicitly,  $C = 2I - A$ , where  $A$  is the adjacency matrix of the underlying graph of  $Q$ .

**1.2. Main results.** Our main result can be stated as follows, where undefined terms will be defined in Section 2.

**Theorem 1.1** (Theorem 3.6, Theorem 4.4). *Let  $\tilde{Q}$  be an extended Dynkin quiver with vertex set  $I = \{0, \dots, n\}$ , where 0 is an extending vertex, and write  $Q$  for the full subquiver obtained by deleting vertex 0. Let  $\lambda \in \mathbb{k}^{n+1}$  be a weight for  $\tilde{Q}$ . Then there exists a subset  $J$  of  $I \setminus \{0\}$  such that, if  $Q^{(1)} \sqcup \dots \sqcup Q^{(r)}$  is the full subquiver of  $Q$  obtained by deleting the vertices in  $J$ , so that the  $Q^{(i)}$  are connected and therefore necessarily Dynkin, there is a triangle equivalence*

$$\mathcal{D}_{\text{sg}}(\mathcal{O}^\lambda(\tilde{Q})) \simeq \bigoplus_{i=1}^r \mathcal{D}_{\text{sg}}(R_{Q^{(i)}}).$$

We can in fact restrict our attention to the case where the weight  $\lambda$  has a particular form, which is most easily stated when  $\mathbb{k} = \mathbb{C}$ . This allows us to give a somewhat more precise result:

**Theorem 1.2** (Theorem 3.6, Theorem 4.4). *Let  $\tilde{Q}$  be an extended Dynkin quiver with vertex set  $I = \{0, \dots, n\}$ , where 0 is an extending vertex, and write  $Q$  for the full subquiver obtained by deleting vertex 0. Let  $\lambda \in \mathbb{C}^{n+1}$  be a weight for  $\tilde{Q}$  with  $\lambda_i \in \{z \in \mathbb{C} \mid \text{Re } z > 0, \text{ or } \text{Re } z = 0 \text{ and } \text{Im } z \geq 0\}$  for  $1 \leq i \leq n$ . Write  $Q_\lambda$  for the full subquiver of  $Q$  with vertex set  $I_\lambda := \{i \in \{1, \dots, n\} \mid \lambda_i = 0\}$ . Suppose that  $Q_\lambda = Q^{(1)} \sqcup \dots \sqcup Q^{(r)}$ , where the  $Q^{(i)}$  are connected and therefore necessarily Dynkin. Then there is a triangle equivalence*

$$\mathcal{D}_{\text{sg}}(\mathcal{O}^\lambda(\tilde{Q})) \simeq \bigoplus_{i=1}^r \mathcal{D}_{\text{sg}}(R_{Q^{(i)}}).$$

This result coincides with the intuition coming from commutative singularity theory which says that deforming a singularity should make it no worse; in our context, deforming a singularity corresponds to making weights at certain vertices of  $\tilde{Q}$  nonzero, and the above theorem says that this makes the singularity category simpler, in some sense.

Now suppose that the weight  $\lambda \in \mathbb{k}^{n+1}$  is given by  $\lambda_0 = 1$  and  $\lambda_i = 0$  for  $1 \leq i \leq n$ , and in

this case write  $\lambda = \varepsilon_0$ . We then consider  $\mathcal{O}^\lambda(\tilde{Q})$  to be a noncommutative analogue of  $R_Q$ . This viewpoint is partially justified by the following immediate corollary:

**Corollary 1.3** (Corollary 4.7). *Retain the notation of Theorem 1.2, and suppose that  $\lambda = \varepsilon_0$ , as above. Then there is a triangle equivalence*

$$\mathcal{D}_{\text{sg}}(\mathcal{O}^\lambda(\tilde{Q})) \simeq \mathcal{D}_{\text{sg}}(R_Q).$$

Our viewpoint that  $\mathcal{O}^\lambda(\tilde{Q})$  is a noncommutative analogue of  $R_Q$  when  $\lambda = \varepsilon_0$  is further supported by the following result, which generalises the geometric McKay correspondence to a noncommutative setting. Here, undefined terms will be defined in Section 5.

**Theorem 1.4** (Theorem 5.12). *Let  $\tilde{Q}$  be an extended Dynkin quiver,  $Q$  the corresponding Dynkin quiver, and  $\lambda = \varepsilon_0$ . Then  $\mathcal{O}^\lambda(\tilde{Q})$  has a noncommutative resolution (which is in fact a deformation), and the exceptional objects in this resolution may be indexed so that the corresponding intersection matrix is  $(-1)$  times  $C$ , where  $C$  is the Cartan matrix of Dynkin type corresponding to  $Q$ .*

The intersection theory used in the above theorem is that of Mori-Smith [MS01]. This result should be seen as a noncommutative analogue of the fact that the matrix whose entries are the intersection multiplicities of the irreducible curves in the exceptional divisor of the minimal resolution of  $\text{Spec } R$  is  $(-1)$  times the Cartan matrix.

We now take a moment to provide an overview of the proof of Theorem 1.2, which is quite lengthy and much of which comes down to a case-by-case analysis. The first important observation to make is that we can restrict our attention to weights  $\lambda$  which are *quasi-dominant*; if  $\mathbb{k} = \mathbb{C}$ , this means that  $\lambda_0$  can be arbitrary, while for  $1 \leq i \leq n$  we require

$$\lambda_i \in \{z \in \mathbb{C} \mid \text{Re } z > 0, \text{ or } \text{Re } z = 0 \text{ and } \text{Im } z \geq 0\}.$$

For all of our calculations, we work in  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$  rather than  $\mathcal{D}_{\text{sg}}(\mathcal{O}^\lambda(\tilde{Q}))$ ; since  $\mathcal{O}^\lambda(\tilde{Q})$  is Gorenstein, these two categories are triangle equivalent by a result of Buchweitz. In Section 3, the restriction to quasi-dominant weights allows us to give a concrete description of  $\mathcal{D}_{\text{sg}}(\mathcal{O}^\lambda(\tilde{Q}))$  as a  $\mathbb{k}$ -linear category in terms of an auxiliary Krull-Schmidt category. We also find that the isoclasses  $V_i$  of indecomposable objects in  $\mathcal{D}_{\text{sg}}(\mathcal{O}^\lambda(\tilde{Q}))$  are indexed by those vertices  $i \geq 1$  with  $\lambda_i = 0$ . This auxiliary category allows us to determine the equivalence of Theorem 1.2, but only as a  $\mathbb{k}$ -linear equivalence.

Determining the triangulated structure is more involved, but a result of Amiot reduces the problem to one of determining the translation functor in  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$  on the  $V_i$ . Write  $\Delta$  for underlying graph of the full subquiver of  $Q$  obtained by deleting those vertices  $i$  with  $\lambda_i \neq 0$ , which will necessarily be a disjoint union of Dynkin graphs. A useful observation is that the translation functor  $\Sigma$  induces a graph automorphism  $\pi$  of  $\Delta$ , in the sense that  $\Sigma V_i = V_{\pi(i)}$ . Since an automorphism of a finite union of Dynkin graphs is uniquely determined by its action on a small number of vertices, we only need to calculate  $\Sigma V_i$  for a small subset of the  $V_i$ , which we also determine.

We now explain how to explicitly calculate  $\Sigma V_i$  for an indecomposable object  $V_i$  of  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$ . We may consider  $V_i$  as an object of  $\text{mod-}R_Q$ , and then, depending on the choice of  $\lambda$ , we identify a particular short exact sequence of  $R_Q$ -modules

$$0 \rightarrow V_i \rightarrow N \rightarrow V_j \rightarrow 0.$$

We may also view this as a sequence of  $\mathcal{O}^\lambda(\tilde{Q})$ -modules, and the sequence is chosen so that it remains exact in  $\text{mod-}\mathcal{O}^\lambda(\tilde{Q})$  and also so that  $N$  is a projective  $\mathcal{O}^\lambda(\tilde{Q})$ -module. This tells us that  $\Sigma V_i = V_j$  in  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$  for this choice of  $\lambda$ , which is enough to complete the proof.

**1.3. Organisation of the paper.** This paper is organised as follows. In Section 2, we recall some basic definitions and facts, and introduce the notation used throughout the paper. In Section 3, the singularity categories of  $\mathcal{O}^\lambda(\tilde{Q})$  are determined as  $\mathbb{k}$ -linear categories. Section 4 is then concerned with determining the triangulated structure provided one knows how the translation functor behaves on certain indecomposable objects. In Section 5 we digress to prove our results on a noncommutative geometric McKay correspondence for a resolution of  $\mathcal{O}^\lambda(\tilde{Q})$  when  $\lambda = \varepsilon_0$ . Section 6 then verifies that the translation functor behaves as claimed in the type  $\mathbb{A}$  case, while Sections 7, 8, 9, and 10 show this in the type  $\mathbb{D}$  and  $\mathbb{E}$  cases.

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## 2. PRELIMINARIES

We now recall some of the definitions and results that we will make use of throughout this paper. In this section,  $R$  will denote an arbitrary ring.

### 2.1. Definitions and basic results.

**Definition 2.1.** A *quiver*  $Q$  is a directed multigraph, and we write  $Q_0$  for the set of vertices and  $Q_1$  for the set of arrows. We equip  $Q$  with head and tail maps  $h, t : Q_1 \rightarrow Q_0$  which take an arrow to the vertices that are its head and tail respectively. A *non-trivial path* in the quiver is a sequence of arrows  $p = \alpha_1 \alpha_2 \dots \alpha_\ell$  with  $h(\alpha_i) = t(\alpha_{i+1})$  for  $1 \leq i \leq \ell - 1$  (that is, we compose arrows from left to right), and such a path is said to have *length*  $\ell$ . Moreover, for each vertex  $i \in Q_0$  there is a *trivial path*  $e_i$  of length 0, with head and tail vertex both equal to  $i$ .

**Definition 2.2.** Given a field  $\mathbb{k}$  and a quiver  $Q$ , we define the path algebra  $\mathbb{k}Q$  of  $Q$  as follows: as a  $\mathbb{k}$ -vector space,  $\mathbb{k}Q$  has a basis given by paths in the quiver, and we define multiplication by concatenation of paths:

$$p \cdot q = \begin{cases} pq & \text{if } h(p) = t(q) \\ 0 & \text{otherwise} \end{cases}.$$

If  $R$  is a commutative ring, then  $\text{Spec } R$  is nonsingular if and only if  $R$  has finite global dimension. It is therefore sensible to say that a (possibly noncommutative) ring is *nonsingular* if it has finite global dimension, and *singular* otherwise. Before we are able to define the singularity category of a ring, we must make a few more definitions.

We write  $\text{mod-}R$  (respectively,  $R\text{-mod}$ ) for the category of finitely generated right (respectively, left)  $R$ -modules; in this paper, we shall use right modules unless stated otherwise. We also write  $\text{proj-}R$  for the full subcategory of  $\text{mod-}R$  consisting of finitely generated projective modules. The *dual* of an  $R$ -module  $M$  is  $M^* := \text{Hom}_R(M, R)$ . We write  $\text{p.dim } M$  and  $\text{i.dim } M$  for the projective and injective dimensions of  $M \in \text{mod-}R$ , respectively, and  $\text{gl.dim } R$  for the global dimension of  $R$ .

**Definition 2.3.** The *stable module category* of  $R$ , denoted  $\underline{\text{mod-}}R$ , is the category whose objects are the same as those of  $\text{mod-}R$ , and for modules  $M, N$ , has morphisms  $\underline{\text{Hom}}_R(M, N) = \text{Hom}_R(M, N)/\sim$ , where  $f \sim f'$  if and only if  $f - f'$  factors through a finitely generated projective module. Given a full subcategory  $\text{abc-}R$  of  $\text{mod-}R$ , we write  $\underline{\text{abc-}}R$  for the full subcategory of  $\underline{\text{mod-}}R$  whose objects are the same as those of  $\text{abc-}R$ .

Noting that an element of  $\sum_{i=1}^k n_i f_i$  of  $NM^*$  may be viewed as a homomorphism  $M \rightarrow N$ , it is not hard to show that a module homomorphism  $f : M \rightarrow N$  factors through a projective module

if and only if  $f \in NM^*$ . This allows us to identify  $\underline{\text{Hom}}_R(M, N)$  with  $\text{Hom}_R(M, N)/NM^*$ , which will be useful in later calculations.

By [AB69, Proposition 1.44], two  $R$ -modules  $M, N$  are isomorphic in  $\underline{\text{mod}}\text{-}R$  if and only if there exist projective modules  $P, Q \in \text{proj-}R$  such that  $M \oplus P \cong N \oplus Q$  in  $\text{mod-}R$ . Recalling that the *first syzygy*  $\Omega M$  of  $M \in \text{mod-}R$  is defined to be the kernel of any surjection  $R^n \twoheadrightarrow M$ , it follows from [Rot08, Proposition 8.5] that  $\Omega M$  is uniquely determined in  $\underline{\text{mod}}\text{-}R$ .

**Definition 2.4.** A finitely generated  $R$ -module  $M$  is said to be *maximal Cohen-Macaulay* (MCM) if it satisfies  $\text{Ext}_R^i(M, R) = 0$  for all  $i \geq 1$ . We write  $\text{MCM-}R$  for the full subcategory of  $\text{mod-}R$  consisting of maximal Cohen-Macaulay  $R$ -modules.

Maximal Cohen-Macaulay modules have the following elementary properties, proofs of which can be found in [Buc86] or [Kal15], for example:

**Lemma 2.5.**

- (1) *Any finitely generated projective module is MCM.*
- (2) *MCM modules are reflexive.*
- (3) *Finite direct sums and direct summands of MCM modules are MCM.*
- (4) *An MCM module is either projective or has infinite projective dimension.*

**Definition 2.6.** A ring  $R$  is said to be *Gorenstein* if it is noetherian (i.e., left and right noetherian) and both  $\text{i.dim } R_R$  and  $\text{i.dim } {}_R R$  are finite. By [Zak69, Lemma A], if  $R$  is noetherian and both  $\text{i.dim } R_R$  and  $\text{i.dim } {}_R R$  are finite then these values coincide, and we call this common value the *(injective) dimension* of  $R$ .

Our reason for introducing these concepts is due to the following result:

**Theorem 2.7** ([Buc86, Theorem 4.4.1]). *The full subcategory  $\underline{\text{MCM-}}R$  of  $\underline{\text{mod}}\text{-}R$  whose objects are MCM  $R$ -modules is a triangulated category, with translation functor  $\Sigma$  given by  $\Sigma M = \Omega^{-1}M$ . Moreover, if  $R$  is Gorenstein then there is a triangle equivalence  $\mathcal{D}_{\text{sg}}(R) \simeq \underline{\text{MCM-}}R$ .*

Justified by the above theorem, we henceforth perform all of our calculations in  $\underline{\text{MCM-}}R$  rather than  $\mathcal{D}_{\text{sg}}(R)$ .

Finally, we recall two useful results that will be helpful when identifying the maximal Cohen-Macaulay modules of a ring. Given an additive category  $\mathcal{C}$  and an object  $C \in \mathcal{C}$ , we write  $\text{add}(C)$  for the full subcategory of  $\mathcal{C}$  consisting of direct summands of finite direct sums of  $C$ . This is the smallest additive subcategory of  $\mathcal{C}$  which contains  $C$  and is closed under taking direct summands. The following result is due to Auslander, but we provide a proof.

**Proposition 2.8** (Auslander). *Suppose that  $R$  is Gorenstein and that  $M \in \text{MCM-}R$  has  $R$  as a direct summand, so in particular is a generator. If  $\text{gl.dim } \text{End}_R(M) \leq 2$ , then  $\text{add } M = \text{MCM-}R$ . Moreover, if  $R$  is of dimension at most 2, then the converse also holds.*

*Proof.* Write  $\Lambda = \text{End}_R(M)$ . We note that the functor  $\text{Hom}_R(M, -) : \text{mod-}R \rightarrow \text{mod-}\Lambda$  restricts to an equivalence

$$\text{add } M \xrightarrow{\sim} \text{proj-}\Lambda. \quad (2.9)$$

( $\Rightarrow$ ) This is [IW10, Proposition 2.11] when  $n = 0$  (note here that the authors are concerned with Cohen-Macaulay modules, but every property of CM modules that is used in the proof is also true for MCM modules).

( $\Leftarrow$ ) Now assume that  $\text{add } M = \text{MCM-}R$  and that  $\text{i.dim } R \leq 2$ . Let  $N \in \text{mod-}\Lambda$ , and consider the initial terms in a projective resolution of  $N$ ,

$$P_1 \xrightarrow{f} P_0 \rightarrow N \rightarrow 0.$$

By (2.9), there exists a morphism  $g : M_1 \rightarrow M_0$  in  $\text{add } M$  with

$$(f : P_1 \rightarrow P_0) = (g \circ - : \text{Hom}_R(M, M_1) \rightarrow \text{Hom}_R(M, M_0))$$

Set  $K = \ker g$ . We have two short exact sequences

$$\begin{aligned} 0 \rightarrow K \rightarrow M_1 \rightarrow \text{im } g \rightarrow 0, \\ 0 \rightarrow \text{im } g \rightarrow M_0 \rightarrow \text{coker } g \rightarrow 0. \end{aligned}$$

Applying  $\text{Hom}_R(-, R)$  to each of these gives rise to exact sequences (here  $i \geq 1$ ),

$$\begin{aligned} \text{Ext}_R^i(M_1, R) \rightarrow \text{Ext}_R^i(K, R) \rightarrow \text{Ext}_R^{i+1}(\text{im } g, R) \rightarrow \text{Ext}_R^{i+1}(M_1, R), \\ \text{Ext}_R^{i+1}(M_0, R) \rightarrow \text{Ext}_R^{i+1}(\text{im } g, R) \rightarrow \text{Ext}_R^{i+2}(\text{coker } g, R). \end{aligned}$$

Note that the flanking terms are all 0: indeed,  $\text{Ext}_R^{i+2}(\text{coker } g, R)$  vanishes since  $\text{i.dim } R \leq 2$ , while the other three terms vanish because  $M_0$  and  $M_1$  are both MCM. Therefore  $\text{Ext}_R^i(K, R) \cong \text{Ext}_R^{i+1}(\text{im } g, R) = 0$  for all  $i \geq 1$ , and so  $K$  is MCM. Since  $\text{add } M = \text{MCM-}R$  by assumption, we have an exact sequence

$$0 \rightarrow K \rightarrow M_1 \rightarrow M_0$$

with each term in  $\text{add } M$ . Then applying  $\text{Hom}_R(M, -)$  gives an exact sequence

$$0 \rightarrow \text{Hom}_R(M, K) \rightarrow P_1 \rightarrow P_0 \rightarrow N \rightarrow 0,$$

where  $\text{Hom}_R(M, K)$  is projective by (2.9). Therefore  $\text{p.dim } N \leq 2$ , and so  $\text{gl.dim } \Lambda \leq 2$ .  $\square$

**Lemma 2.10.** *Let  $R$  be a Gorenstein ring of dimension at most 2. Then  $M \in \text{mod-}R$  is reflexive if and only if it is maximal Cohen-Macaulay.*

*Proof.* ( $\Leftarrow$ ) This is Lemma 2.5 (2).

( $\Rightarrow$ ) Suppose now that  $M$  is reflexive. Since  $R$  is noetherian,  $M^*$  is finitely presented, so we have an exact sequence of the form

$$R^{\oplus m} \rightarrow R^{\oplus n} \rightarrow M^* \rightarrow 0.$$

Applying  $\text{Hom}_R(-, R)$  and noting that  $M$  is reflexive yields an exact sequence

$$0 \rightarrow M \rightarrow R^{\oplus n} \xrightarrow{\theta} R^{\oplus m} \rightarrow \text{coker } \theta \rightarrow 0.$$

But then, by [Rot08, Corollary 6.55],  $\text{Ext}_R^i(M, R) \cong \text{Ext}_R^{i+2}(\text{coker } \theta, R) = 0$  for all  $i \geq 1$ , where the last equality follows since  $\text{i.dim } R \leq 2$ . That is,  $M$  is maximal Cohen-Macaulay.  $\square$

**2.2. The deformations of Crawley-Boevey and Holland.** In [CBH98], Crawley-Boevey and Holland introduced the notion of the *deformed preprojective algebra* of a quiver  $Q$ , and, if  $\tilde{Q}$  is extended Dynkin, a family of  $\mathbb{k}$ -algebras  $\mathcal{O}^\lambda(\tilde{Q})$  which may be thought of as deformations of the coordinate ring of a Kleinian singularity. We now recall these definitions, noting that our definition of  $\mathcal{O}^\lambda(\tilde{Q})$  differs slightly from that of Crawley-Boevey and Holland, but is consistent with theirs by [CBH98, Theorem 0.1].

**Definition 2.11.** Let  $Q$  be a quiver without loops. The *double* of  $Q$  is the quiver  $\overline{Q}$  obtained from  $Q$  by adding a *reverse arrow*  $\overline{\alpha} : j \rightarrow i$  for each arrow  $\alpha : i \rightarrow j$  in  $Q$ . We call the arrows in  $\overline{Q}$  which are not reverse arrows *ordinary arrows*. Given a *weight*  $\lambda \in \mathbb{k}^{Q_0}$  for  $Q$ , the corresponding *deformed preprojective algebra* is the  $\mathbb{k}$ -algebra

$$\Pi^\lambda(Q) := \mathbb{k}\overline{Q}/I$$

where  $I$  is the two-sided ideal of  $\mathbb{k}\tilde{Q}$  with generators

$$\sum_{\substack{\alpha \in Q_0 \\ t(\alpha)=i}} \alpha \bar{\alpha} - \sum_{\substack{\alpha \in Q_0 \\ h(\alpha)=i}} \bar{\alpha} \alpha - \lambda_i e_i$$

for each vertex  $i \in Q_0$ .

It is helpful to think of a weight as label from  $\mathbb{k}$  at each vertex of  $Q$ , and we will usually display it this way in diagrams. We will often refer to  $\lambda_i$  as the weight at vertex  $i$ .

Now suppose that  $\tilde{Q}$  is extended Dynkin, with vertices and edges (of its double) labelled as in Figure 1. (Observe that, other than in type  $\mathbb{A}$ , the arrows pointing towards a vertex all have the same orientation, and so any path in the double of  $\tilde{Q}$  alternates between ordinary arrows and reverse arrows.) Throughout this paper, it will be our convention that  $\tilde{Q}$  denotes an extended Dynkin quiver, while  $Q$  will denote the corresponding Dynkin quiver obtained by removing the extending vertex  $0$ <sup>1</sup>, where the orientation of the arrows comes from Figure 1.

We are now able to define the algebras of interest to us. They are given by

$$\mathcal{O}^\lambda(\tilde{Q}) := e_0 \Pi^\lambda(\tilde{Q}) e_0.$$

The elements of  $\mathcal{O}^\lambda(\tilde{Q})$  may be thought of as (equivalence classes of) paths in the double of  $\tilde{Q}$  which start and end at the extending vertex  $0$ .

If  $\lambda = \mathbf{0}$ , then  $\Pi(Q) := \Pi^\lambda(Q)$  is the (unweighted) preprojective algebra of Gelfand and Ponomarev [GP79], and in this case we also write  $\mathcal{O}(\tilde{Q}) := \mathcal{O}^\lambda(\tilde{Q})$ . We will often write  $\Pi^\lambda$  and  $\mathcal{O}^\lambda$  (or  $\Pi$  and  $\mathcal{O}$  if  $\lambda = \mathbf{0}$ ) when the corresponding quiver is either unimportant or understood.

For our purposes, it is important to know precisely when the rings  $\mathcal{O}^\lambda$  are noncommutative. This depends on the weight  $\lambda$  and also on a vector  $\delta \in \mathbb{N}^{\tilde{Q}_0}$ , which we now define. Let  $G$  be the finite subgroup of  $\mathrm{SL}(2, \mathbb{k})$  corresponding to  $\tilde{Q}$ . Then, under the McKay correspondence, each vertex of  $\tilde{Q}$  corresponds to an irreducible representation  $W_i$  of  $G$ , and we set  $\delta_i := \dim_{\mathbb{k}} W_i$ . If we number the vertices of  $\tilde{Q}$  as in Figure 1, then

$$\begin{aligned} \tilde{\mathbb{A}}_n : \quad \delta &= (\underbrace{1, 1, \dots, 1}_{n+1 \text{ times}}, 1) \\ \tilde{\mathbb{D}}_n : \quad \delta &= (1, 1, \underbrace{2, 2, \dots, 2}_{n-3 \text{ times}}, 2, 1, 1) \\ \tilde{\mathbb{E}}_6 : \quad \delta &= (1, 2, 1, 2, 3, 2, 1) \\ \tilde{\mathbb{E}}_7 : \quad \delta &= (1, 2, 3, 4, 3, 2, 1, 2) \\ \tilde{\mathbb{E}}_8 : \quad \delta &= (1, 2, 3, 4, 5, 6, 4, 2, 3). \end{aligned}$$

The first part of the next result is Theorem 0.4 (1) of [CBH98], while the second part is obtained by combining Theorem 0.1 and Lemma 1.1 of [loc. cit.]:

**Theorem 2.12.**  *$\mathcal{O}^\lambda$  is commutative if and only if  $\lambda \cdot \delta = \sum_{i \in \tilde{Q}_0} \lambda_i \delta_i = 0$ . In the case when  $\lambda = \mathbf{0}$ ,  $\mathcal{O}$  is isomorphic to the coordinate ring of the Kleinian singularity corresponding to  $\tilde{Q}$ .*

In fact, by [CBH98, Lemma 2.2], one may always assume that  $\lambda \cdot \delta$  is either 0 or 1. If  $\lambda$  is a weight with  $\lambda \cdot \delta = 0$ , so that  $\mathcal{O}^\lambda$  is commutative, then if we define  $\lambda' = (\lambda_0 + 1, \lambda_1, \dots, \lambda_n)$ , we may consider  $\mathcal{O}^{\lambda'}$  to be a noncommutative analogue of  $\mathcal{O}^\lambda$ . One of the main questions that

<sup>1</sup>It is common to refer to any vertex  $i$  with  $\delta_i = 1$  as an extending vertex, but for us the extending vertex is vertex  $0$  in Figure 1.

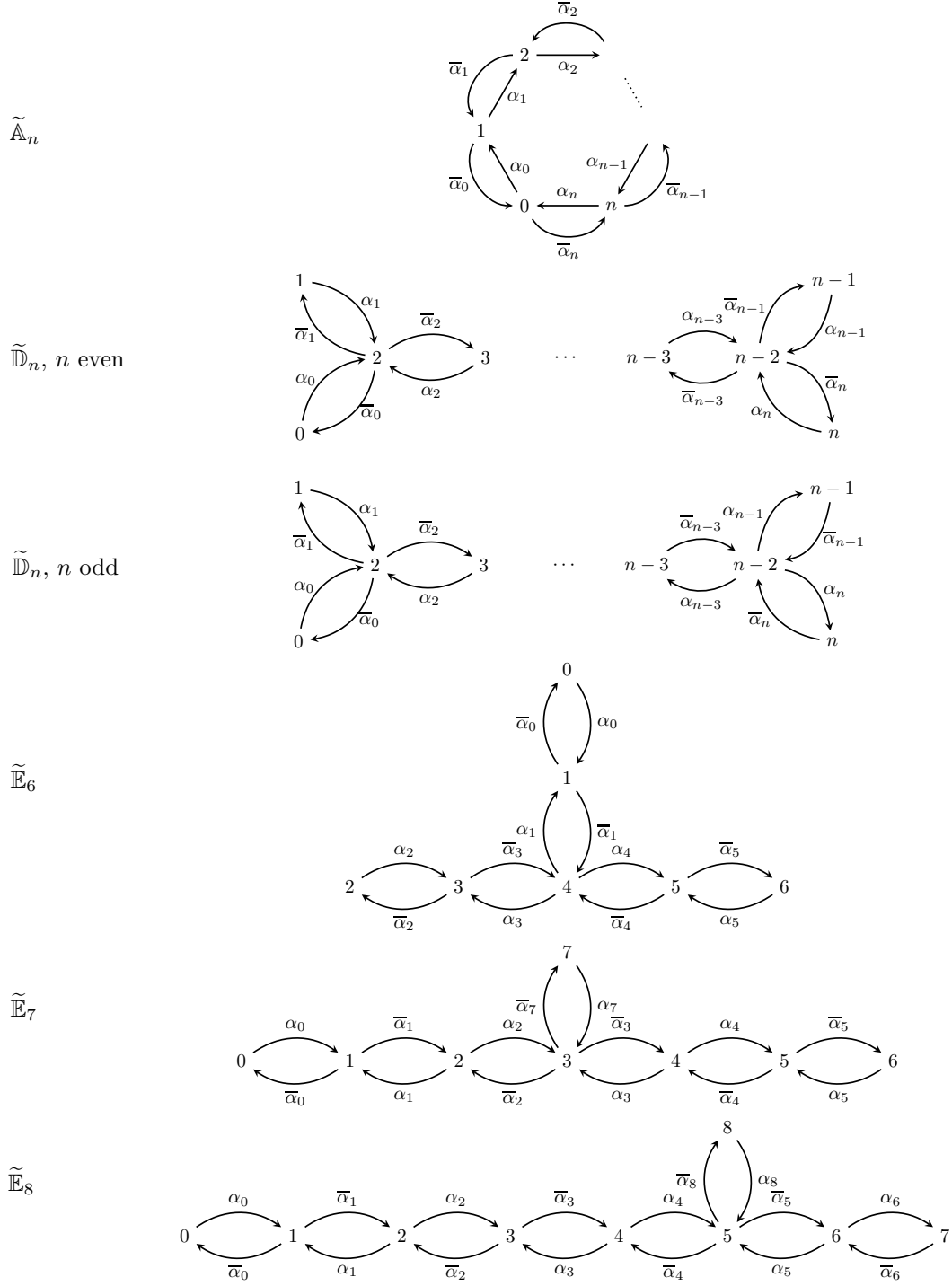


FIGURE 1. Labellings of the vertices and arrows of the doubles of extended Dynkin quivers.



we set out to answer was: what is the relationship between the singularity categories of  $\mathcal{O}^\lambda$  and  $\mathcal{O}^{\lambda'}$ ? In fact, it follows from Theorem 1.2 that the singularity categories of these  $\mathbb{k}$ -algebras are triangle equivalent.

In [CBH98, §7], the authors prove a number of results in the case where the weight  $\lambda$  is *dominant*, a term which we now define. Fix a total ordering  $\prec$  on  $\mathbb{k}$  which also satisfies the following:

- (1) If  $a \prec b$ , then  $a + c \prec b + c$  for all  $c \in \mathbb{k}$ ;
- (2) On the integers,  $\prec$  coincides with the usual order;
- (3) For any  $a \in \mathbb{k}$ , there exists  $m \in \mathbb{Z}$  with  $a \prec m$ .

For example, when  $\mathbb{k} = \mathbb{C}$  we may define  $\prec$  by  $z \prec z'$  if and only if  $\operatorname{Re} z < \operatorname{Re} z'$ , or  $\operatorname{Re} z = \operatorname{Re} z'$  and  $\operatorname{Im} z \prec \operatorname{Im} z'$ . We then say that a weight  $\lambda \in \mathbb{k}^{Q_0}$  is *dominant* if  $\lambda_i \succeq 0$  for all  $i \in Q_0$ . In the case where the quiver  $\tilde{Q}$  is extended Dynkin, we say that  $\lambda$  is *quasi-dominant* if  $\lambda_i \succeq 0$  for all  $i \neq 0$ .

By combining Lemmas 7.8 and 7.9 of [CBH98], it follows that given a weight  $\lambda$  for an extended Dynkin quiver  $\tilde{Q}$ , there exists a quasi-dominant weight  $\lambda'$  such that  $\mathcal{O}^\lambda$  and  $\mathcal{O}^{\lambda'}$  are Morita equivalent. In fact, unpublished work of Boddington and Levy [BL07] shows that Lemma 7.9 of [CBH98] can be strengthened as follows:

**Lemma 2.13.** *Suppose that  $\lambda$  is a weight for an extended Dynkin quiver  $\tilde{Q}$ , and let  $\rho$  be a sequence of a dual reflections (defined in Section 5.1) at vertices other than the extending vertex 0. Then  $\mathcal{O}^\lambda \cong \mathcal{O}^{\rho(\lambda)}$ .*

Combining this with Lemma 7.8 of [CBH98] establishes the following:

**Lemma 2.14.** *Suppose that  $\lambda$  is a weight for an extended Dynkin quiver  $\tilde{Q}$ . Then there exists a quasi-dominant weight  $\lambda'$  with  $\mathcal{O}^\lambda \cong \mathcal{O}^{\lambda'}$ .*

If we restrict our attention to quasi-dominant weights then it is easy to detect whether  $\mathcal{O}^\lambda$  is singular:

**Lemma 2.15.** *If  $\lambda$  is a quasi-dominant weight for an extended Dynkin quiver  $\tilde{Q}$ , then  $\mathcal{O}^\lambda$  is singular if and only if  $\lambda_i = 0$  for some  $i \neq 0$ .*

*Proof.* Suppose that  $\tilde{Q}$  has  $n + 1$  vertices. By [CBH98, Theorem 0.4 (4)],  $\mathcal{O}^\lambda$  is singular if and only if  $\lambda \cdot \alpha = 0$  for some Dynkin root  $\alpha$ . The possible values of these so-called Dynkin roots are not important to us; it suffices to know that they have the form  $(0, \alpha') \in \mathbb{Z}^{n+1}$  where, in particular,  $\alpha'$  has entirely non-negative or non-positive entries, and at least one nonzero entry. In addition,  $\varepsilon_i \in \mathbb{Z}^{n+1}$  for  $1 \leq i \leq n$  is always a Dynkin root, where  $\varepsilon_i$  is the  $i^{\text{th}}$  coordinate vector (here the entries are indexed from 0 to  $n$ ). Therefore, if  $\lambda_i = 0$  for some  $i \neq 0$  then  $\lambda \cdot \alpha = 0$  for the Dynkin root  $\alpha = \varepsilon_i$ , while if  $\lambda_i \neq 0$  for all  $i \neq 0$ , then necessarily  $\lambda \cdot \alpha \neq 0$  for all Dynkin roots  $\alpha$ . The result then follows.  $\square$

Restricting our attention to quasi-dominant weights will frequently be useful, and so using Lemma 2.14 we assume that this is the case for the remainder of the paper:

**Assumption 2.16.** *If  $\lambda$  is a weight for an extended Dynkin quiver  $\tilde{Q}$ , then we always assume that the weight  $\lambda$  is quasi-dominant.*

We will see later that this assumption allows one to easily read off a number of useful facts about the module category of  $\mathcal{O}^\lambda$ , and ultimately its singularity category as well.

### 3. THE SINGULARITY CATEGORY OF $\mathcal{O}^\lambda(\tilde{Q})$ AS A $\mathbb{k}$ -LINEAR CATEGORY

The first step in determining  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$  as a triangulated category is to determine its structure as an additive category, or indeed as a  $\mathbb{k}$ -linear category. We first identify an important module.

**Lemma 3.1.**  $\Pi^\lambda e_0$  is a finitely generated  $\mathcal{O}^\lambda$ -module, and it satisfies  $\text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \Pi^\lambda$ .

*Proof.* Since  $\Pi^\lambda$  is noetherian,  $\Pi^\lambda e_0 \Pi^\lambda$  is a finitely generated ideal of  $\Pi^\lambda$ , and so [MS81, Lemma 1] ensures that  $\Pi^\lambda e_0$  is a finitely generated  $\mathcal{O}^\lambda$ -module. For the second claim, it is clear that we have an inclusion  $\Pi^\lambda \subseteq \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0)$ , where an element of  $\Pi^\lambda$  gives rise to an endomorphism of  $\Pi^\lambda e_0$  by left multiplication. Filtering  $\Pi^\lambda$  and  $\mathcal{O}^\lambda$  by path length, [CBH98, Lemma 1.1, Corollary 3.6] tells us that we have

$$\Pi = \text{gr } \Pi^\lambda \subseteq \text{gr } \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \subseteq \text{End}_{\text{gr } \mathcal{O}^\lambda}(\text{gr } \Pi^\lambda e_0) = \text{End}_{\mathcal{O}}(\Pi e_0) = \Pi.$$

Therefore we have both  $\Pi^\lambda \subseteq \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0)$  and  $\text{gr } \Pi^\lambda = \text{gr } \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0)$  and it is standard (c.f. [MR01, Proposition 1.6.7]) that together these facts imply  $\Pi^\lambda = \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0)$ .  $\square$

Write  $V_i = e_i \Pi^\lambda e_0$ ; we shall refer to these  $\mathcal{O}^\lambda$ -modules as *vertex modules*, and these will play an important role in determining  $\text{MCM-}\mathcal{O}^\lambda$ .

As a corollary of Lemma 3.1, we are able to calculate the Hom spaces between the vertex modules.

**Corollary 3.2.** We have  $\text{Hom}_{\mathcal{O}^\lambda}(V_i, V_j) = e_j \Pi^\lambda e_i$ , and so  $\Pi^\lambda e_0$  is a reflexive (and hence maximal Cohen-Macaulay)  $\mathcal{O}^\lambda$ -module.

*Proof.* By Lemma 3.1,  $\Pi^\lambda = \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \bigoplus_{k,\ell} \text{Hom}_{\mathcal{O}^\lambda}(e_k \Pi^\lambda e_0, e_\ell \Pi^\lambda e_0)$ . Multiplying on the left by  $e_j$  kills each Hom space with  $\ell \neq j$ , while multiplying on the right by  $e_i$  kills those Hom spaces with  $k \neq i$ . It follows that

$$e_j \Pi^\lambda e_i = e_j \left( \bigoplus_{k,\ell} \text{Hom}_{\mathcal{O}^\lambda}(V_k, V_\ell) \right) e_i = e_j \text{Hom}_{\mathcal{O}^\lambda}(V_i, V_j) e_i = \text{Hom}_{\mathcal{O}^\lambda}(V_i, V_j).$$

Since all of our results on Hom spaces are left-right symmetrical, we also find that

$$\begin{aligned} (\Pi^\lambda e_0)^{**} &= \left( \text{Hom}_{\mathcal{O}^\lambda} \left( \bigoplus_{i=0}^n V_i, V_0 \right) \right)^* = \left( \bigoplus_{i=0}^n e_0 \Pi^\lambda e_i \right)^* \\ &= \bigoplus_{i=0}^n \text{Hom}_{\mathcal{O}^\lambda}(e_0 \Pi^\lambda e_i, e_0 \Pi^\lambda e_0) = \bigoplus_{i=0}^n V_i = \Pi^\lambda e_0, \end{aligned}$$

and so  $\Pi^\lambda e_0$  is a reflexive  $\mathcal{O}^\lambda$ -module. Therefore, since  $\Pi^\lambda e_0$  is finitely generated, and since  $\text{i.dim } \mathcal{O}^\lambda \leq 2$  ([CBH98, Theorem 1.6]), it follows that  $\Pi^\lambda e_0$  is maximal Cohen-Macaulay.  $\square$

We are also able to determine when the vertex modules are projective. This is the first instance of the quasi-dominance of  $\lambda$  revealing important information about  $\text{mod-}\mathcal{O}^\lambda$ .

**Lemma 3.3.** If  $i = 0$  or  $\lambda_i \neq 0$ , then  $V_i$  is a projective  $\mathcal{O}^\lambda$ -module.

*Proof.* When  $i = 0$  this is clear. So suppose that  $i \neq 0$  and  $\lambda_i \neq 0$ . Then, using [CBH98, Lemma 7.1 (1)],  $e_i = 0$  in  $\Pi^\lambda / \Pi^\lambda e_0 \Pi^\lambda$  and so  $e_i \in \Pi^\lambda e_0 \Pi^\lambda$ . But then, using Corollary 3.2,

$$V_i V_i^* = e_i \Pi^\lambda e_0 \Pi^\lambda e_i \ni e_i^3 = e_i,$$

where  $e_i$  is the identity element of  $\text{End}_{\Pi^\lambda}(V_i) = e_i \Pi^\lambda e_i$ , and so  $V_i$  is projective by the dual basis lemma.  $\square$

It follows that the vertex modules whose weight is nonzero are equal to the zero object in the singularity category. When working stably, we will sometimes refer to those vertex modules whose corresponding weight is zero as non-projective vertex modules.

We recall some notation from the introduction which will be used throughout the rest of the

paper. Write  $Q_\lambda$  for the full subquiver of  $Q$  with vertex set  $I_\lambda := \{i \in \{1, \dots, n\} \mid \lambda_i = 0\}$ . By the above, the vertices of  $Q_\lambda$  correspond precisely to those vertex modules which are not projective.

**Lemma 3.4.** *We have  $\underline{\text{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \cong \Pi(Q_\lambda)$ .*

*Proof.* Write  $\mu = (\lambda_1, \dots, \lambda_n)$ . By Corollary 3.2, we have that

$$(\Pi^\lambda e_0)^* = \bigoplus_i \text{Hom}_{\mathcal{O}^\lambda}(e_i \Pi^\lambda e_0, e_0 \Pi^\lambda e_0) = \bigoplus_i e_0 \Pi^\lambda e_i = e_0 \Pi^\lambda.$$

We now claim that  $\Pi^\lambda / \Pi^\lambda e_0 \Pi^\lambda \cong \Pi^\mu(Q)$ . We can define a ring homomorphism

$$\phi : \mathbb{k}\tilde{Q} \rightarrow \Pi^\mu(Q), \quad \phi(p) = \begin{cases} 0 & \text{if } p \text{ goes through vertex } 0 \\ p & \text{otherwise} \end{cases}.$$

Now suppose that  $0 \leq i \leq n$ , and consider the element

$$p = \sum_{\substack{\alpha \in Q_0 \\ t(\alpha)=i}} \alpha \bar{\alpha} - \sum_{\substack{\alpha \in Q_0 \\ h(\alpha)=i}} \bar{\alpha} \alpha - \lambda_i e_i \in \mathbb{k}\tilde{Q}.$$

If  $i = 0$  then clearly  $\phi(p) = 0$ , while if  $i \neq 0$  then we also have  $\phi(p) = 0$  since  $\mu_i = \lambda_i$  and  $p$  is one of the generators of the ideal of relations for  $\Pi^\mu(Q)$ . That is, every generator of the ideal of relations for  $\Pi^\lambda(\tilde{Q})$  lies in the kernel of  $\phi$ , so  $\phi$  induces a map  $\tilde{\phi} : \Pi^\lambda(\tilde{Q}) \rightarrow \Pi^\mu(Q)$ . But then the kernel of  $\tilde{\phi}$  is simply those paths which go through the extending vertex, namely  $\Pi^\lambda(\tilde{Q})e_0\Pi^\lambda(\tilde{Q})$ , and the claim follows.

Now noting that  $\Pi^\lambda e_0 (\Pi^\lambda e_0)^* = \Pi^\lambda e_0 \Pi^\lambda$ , we have

$$\underline{\text{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \cong \frac{\Pi^\lambda}{\Pi^\lambda e_0 \Pi^\lambda} \cong \Pi^\mu(Q) \cong \Pi(Q_\lambda),$$

where the last isomorphism follows from [CBH98, Lemma 7.1 (1)].  $\square$

**Proposition 3.5.**

- (1) In  $\text{MCM-}\mathcal{O}^\lambda$  we have  $\text{MCM-}\mathcal{O}^\lambda = \text{add } \Pi^\lambda e_0$ .
- (2) In  $\underline{\text{MCM-}}\mathcal{O}^\lambda$  we have  $\underline{\text{MCM-}}\mathcal{O}^\lambda = \text{add} \left( \bigoplus_{i \in I_\lambda} V_i \right)$ .

*Proof.*

- (1) First note that  $\mathcal{O}^\lambda$  is Gorenstein and that, using [CBH98, Theorem 1.5.],

$$\text{gl.dim } \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \text{gl.dim } \Pi^\lambda \leq 2.$$

Since  $\Pi^\lambda e_0$  has  $\mathcal{O}^\lambda$  as a direct summand, the first claim then follows from Proposition 2.8.

- (2) Part (1) immediately implies that  $\underline{\text{MCM-}}\mathcal{O}^\lambda = \text{add}_{\underline{\text{MCM-}}\mathcal{O}^\lambda} \Pi^\lambda e_0 = \text{add}_{\underline{\text{MCM-}}\mathcal{O}^\lambda} \left( \bigoplus_i V_i \right)$ . But projective modules are killed when passing to the stable module category, so the result follows by Lemma 3.3.  $\square$

**Theorem 3.6.** *The functor  $\underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -)$  induces a  $\mathbb{k}$ -linear equivalence*

$$\underline{\text{MCM-}}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(Q_\lambda).$$

*Proof.* By [Kra14, Proposition 2.3], the functor

$$\underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -) : \underline{\text{mod-}}\mathcal{O}^\lambda \rightarrow \text{mod-}\underline{\text{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \text{mod-}\Pi(Q_\lambda)$$

induces a fully faithful  $\mathbb{k}$ -linear functor  $\text{add } \Pi^\lambda e_0 \rightarrow \text{proj-}\Pi(Q_\lambda)$ , where  $\text{add } \Pi^\lambda e_0 = \underline{\text{MCM-}}\mathcal{O}^\lambda$  by (the proof of) Proposition 3.5. Now,  $\text{proj-}\Pi(Q_\lambda) = \text{add}(\bigoplus_i e_i \Pi(Q_\lambda))$  and

$$\underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, V_i) = \frac{e_i \Pi^\lambda}{e_i \Pi^\lambda e_0 \Pi^\lambda} = \frac{e_i \Pi^\lambda}{e_i \Pi^\lambda \cap \Pi^\lambda e_0 \Pi^\lambda} = e_i \frac{\Pi^\lambda}{\Pi^\lambda e_0 \Pi^\lambda} = e_i \Pi(Q_\lambda).$$

Therefore the functor is also essentially surjective, and so we have the claimed equivalence.  $\square$

This is quite an intuitive result. In Lemma 2.15 we saw that  $\mathcal{O}^\lambda$  is singular if and only if  $\lambda_i = 0$  for some  $i \neq 0$ , and this happens precisely when  $\text{proj-}\Pi(Q_\lambda) \simeq \underline{\text{MCM-}}\mathcal{O}^\lambda$  is nontrivial. In particular, the singularity categories of the deformations  $\mathcal{O}^\lambda$  are nontrivial if and only if  $\mathcal{O}^\lambda$  is singular, as one would expect. Moreover, the vertex modules  $V_i$  with  $i = 0$ , or with  $i \neq 0$  and  $\lambda_i \neq 0$ , are the vertex modules which are projective, and hence vanish in  $\underline{\text{MCM-}}\mathcal{O}^\lambda$ , and this is reflected by the fact that these are the vertices which are deleted to obtain  $Q_\lambda$ .

We recall that an additive category is said to be *Krull-Schmidt* if every object decomposes into a finite direct sum of objects, each of which has a local endomorphism ring, and where this decomposition is essentially unique. As an immediate consequence of Theorem 3.6, we have the following result:

**Corollary 3.7.**  *$\underline{\text{MCM-}}\mathcal{O}^\lambda$  is a Krull-Schmidt category.*

*Proof.* Since  $\Pi(Q_\lambda)$  is finite-dimensional [BES07, Proposition 2.1],  $\text{mod-}\Pi(Q_\lambda)$  is Krull-Schmidt and hence so too is  $\text{proj-}\Pi(Q_\lambda)$ . But this is a property which is preserved under  $\mathbb{k}$ -linear equivalence, and so we are done by Theorem 3.6.  $\square$

*Remark 3.8.*

(1) By Proposition 3.5, the objects of  $\underline{\text{MCM-}}\mathcal{O}^\lambda$  are direct summands of finite direct sums of the non-projective vertex modules. Since these vertex modules are indecomposable and  $\underline{\text{MCM-}}\mathcal{O}^\lambda$  is Krull-Schmidt, in fact every object of  $\underline{\text{MCM-}}\mathcal{O}^\lambda$  is isomorphic to a finite direct sum of vertex modules.

(2) We now explain what the equivalence of Theorem 3.6 does to morphisms, and by (1), it suffices to consider morphisms  $V_i \rightarrow V_j$ , which are spanned by the morphisms given by left multiplication by a path  $p$  from vertex  $j$  to vertex  $i$ . This is sent to the morphism  $e_i \Pi(Q_\lambda) \rightarrow e_j \Pi(Q_\lambda)$  given by left multiplication by the path  $p$  if all of the arrows in  $p$  are still present in the double quiver of  $Q_\lambda$ , and is sent to the zero map otherwise. This corresponds to the fact that, when passing to the stable category, a morphism between vertex modules is zero if and only if it factors through a projective module, which happens if and only if the corresponding path goes through the zero vertex or a vertex with a non-zero weight, which happens if and only if some of the arrows of the path are no longer present in the double of  $Q_\lambda$ .

It is possible at this point to say something more. First, an easy lemma:

**Lemma 3.9.** *Let  $Q = Q^{(1)} \sqcup \dots \sqcup Q^{(r)}$  be a finite quiver which is a disjoint union of connected quivers  $Q^{(i)}$ . Then there is a  $\mathbb{k}$ -linear equivalence*

$$\text{proj-}\Pi(Q) \simeq \bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)}).$$

*Proof.* We prove only the case where  $Q$  has two connected components,  $Q^{(1)}$  and  $Q^{(2)}$ , with the general result following by induction. First note that  $\Pi(Q) \cong \Pi(Q^{(1)}) \times \Pi(Q^{(2)})$  via the map

$$\phi : \Pi(Q) \rightarrow \Pi(Q^{(1)}) \times \Pi(Q^{(2)}), \quad \phi(p) = (e^{(1)}p, e^{(2)}p),$$

where  $e^{(i)}$  is the sum of the vertex idempotents in  $Q^{(i)}$ . The result then follows since for any rings  $R$  and  $S$  there is an equivalence  $\text{proj-}(R \times S) \simeq \text{proj-}R \oplus \text{proj-}S$ .  $\square$

The following two corollaries are then immediate from Lemma 3.9.

**Corollary 3.10.** *Suppose that  $\tilde{Q}$  is an extended Dynkin quiver and  $Q_\lambda = Q^{(1)} \sqcup \dots \sqcup Q^{(r)}$  is a disjoint union of connected quivers  $Q^{(i)}$ , which are therefore necessarily Dynkin. Then there is a  $\mathbb{k}$ -linear equivalence*

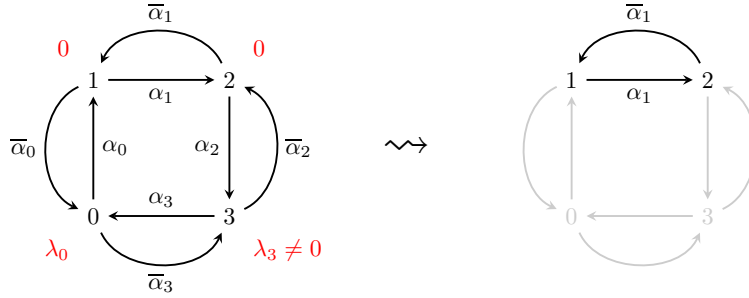
$$\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)}). \quad \square$$

**Corollary 3.11.** *Let  $\tilde{Q}$  and  $\tilde{Q}'$  be extended Dynkin quivers (not necessarily of the same type) and let  $\lambda$  and  $\lambda'$  be quasi-dominant weights for  $\tilde{Q}$  and  $\tilde{Q}'$ , respectively. Suppose that  $Q_\lambda$  and  $Q_{\lambda'}$  are equal to the same disjoint union of Dynkin quivers,  $Q^{(1)} \sqcup \dots \sqcup Q^{(r)}$ . Then there is a  $\mathbb{k}$ -linear equivalence*

$$\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q}) \simeq \underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda'}(\tilde{Q}'). \quad \square$$

*Remark 3.12.* The equivalences arising from Corollary 3.11 are not unique. In general, there are at least as many equivalences between  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$  and  $\underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda'}(\tilde{Q}')$  as there are graph automorphisms of  $Q_\lambda (= Q_{\lambda'})$ . However, if  $\tilde{Q} = \tilde{Q}'$  and, for  $i \neq 0$ ,  $\lambda_i = 0$  precisely when  $\lambda'_i = 0$ , then the obvious equivalence sends a non-projective vertex module  $V_i$  in  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$  to the corresponding (necessarily nonzero) vertex module in  $\underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda'}(\tilde{Q}')$ .

*Example 3.13.* Suppose that  $\tilde{Q} = \mathbb{A}_3$ , and consider the algebra  $\mathcal{O}^\lambda$  where the weight  $\lambda$  is indicated in red on the left hand quiver below, and where  $\lambda_3 \neq 0$ :



By Theorem 3.6 there is a  $\mathbb{k}$ -linear equivalence  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(Q_\lambda)$ , where  $Q_\lambda$  is the subquiver with vertex set  $\{1, 2\}$ . If we set  $\lambda_0 = -\lambda_3$  (respectively,  $\lambda_0 = 1 - \lambda_3$ ) and write  $\lambda^c$  (respectively,  $\lambda^{\text{nc}}$ ) for the resulting weight, then  $\mathcal{O}^{\lambda^c}$  is commutative while  $\mathcal{O}^{\lambda^{\text{nc}}}$  is noncommutative. Write  $F_c$  and  $F_{\text{nc}}$  for the equivalences of Theorem 3.6 in each of these respective cases. Then Corollary 3.11 tells us that we have a  $\mathbb{k}$ -linear equivalence  $\underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda^c} \simeq \underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda^{\text{nc}}}$ .  $F_{\text{nc}}^{-1} \circ F_c$  is one such equivalence, and this sends  $V_i \mapsto V_i$  ( $i = 1, 2$ ),  $\alpha_1 \mapsto \alpha_1$ , and  $\bar{\alpha}_1 \mapsto \bar{\alpha}_1$ . However, one could have just as well considered the equivalence  $F_{\text{nc}}^{-1} \circ \nu \circ F_c$ , where  $\nu$  is the Nakayama automorphism of  $\text{proj-}\Pi(\mathbb{A}_2)$  (see Proposition 4.2), which sends  $V_1 \mapsto V_2$ ,  $V_2 \mapsto V_1$ ,  $\alpha_1 \mapsto \bar{\alpha}_1$ , and  $\bar{\alpha}_1 \mapsto \alpha_1$ .

It is easy to describe  $\mathcal{O}^{\lambda^c}$  and  $\mathcal{O}^{\lambda^{\text{nc}}}$  in terms of generators and relations, and then we can write the above equivalence as

$$\underline{\text{MCM}}\text{-}\frac{\mathbb{k}[x, y, z]}{\langle xy - z^3(z + \lambda_3) \rangle} \simeq \underline{\text{MCM}}\text{-}\frac{\mathbb{k}\langle x, y, z \rangle}{\left\langle \begin{array}{ll} xz = (z + 1)x, & xy = z^3(z + \lambda_3) \\ yz = (z - 1)y, & yx = (z - 1)^3(z + \lambda_3 - 1) \end{array} \right\rangle}.$$

The ring appearing on the right hand side here is an example of a generalised Weyl algebra, as studied in [Bav92, Hod93].

The  $\mathbb{k}$ -linear equivalences of Theorem 3.6 and Corollary 3.10 induce a triangulated structure on  $\text{proj-}\Pi(Q_\lambda) \simeq \bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$ . By [Ami07, §9] there is a canonical triangulated structure on each of the  $\text{proj-}\Pi(Q^{(i)})$  and hence on  $\text{proj-}\Pi(Q_\lambda)$ , and we next wish to verify that this coincides with the induced triangulated structure on  $\text{proj-}\Pi(Q_\lambda)$  from the equivalence of Theorem 3.6. This will also ensure that the equivalence of Corollary 3.11 is actually a triangle equivalence.

#### 4. THE SINGULARITY CATEGORY OF $\mathcal{O}^\lambda(\tilde{Q})$ AS A TRIANGULATED CATEGORY

We retain the notation of the previous section. Since the singularity category of a Gorenstein ring  $R$  is triangulated, it is essential to ask whether the  $\mathbb{k}$ -linear equivalence of Corollary 3.11 is actually a triangle equivalence.

The triangulated structure of  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$  was first described by Buchweitz in [Buc86]. Alternatively,  $\mathcal{O}^\lambda$  is Gorenstein and so the category  $\text{MCM-}\mathcal{O}^\lambda$  is Frobenius (where the projective MCM modules are the projective-injective objects), so the same triangulated structure on  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$  can be obtained by a result of Happel [Hap88, I.2.6]. The translation functor  $\Sigma$  is given by the inverse of the syzygy functor  $\Omega$ ; in fact, we shall see that in our setting  $\Sigma^2 = \text{id}$  and so  $\Sigma = \Omega$ . The distinguished triangles are then defined as follows. Let  $f : X \rightarrow Y$  be a morphism in  $\text{MCM-}\mathcal{O}^\lambda$ . As in [Hap88, 2.2], we can embed  $X$  into a projective (and hence MCM) module  $P$ , and this (by definition) determines a short exact sequence  $0 \rightarrow X \xrightarrow{i} P \rightarrow \Sigma X \rightarrow 0$ . Let  $Z$  be the pushout of  $f$  and  $i$  so that there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & P & \longrightarrow & \Sigma X \longrightarrow 0 \\ & & \downarrow f & & \downarrow & & \parallel \\ 0 & \longrightarrow & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \longrightarrow 0 \end{array}$$

We then call

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h} \Sigma X, \quad (4.1)$$

a typical triangle, and any triangle which is isomorphic to a typical triangle is distinguished.

In [Ami07, §9] it is shown that, if  $Q$  is Dynkin, then  $\text{proj-}\Pi(Q)$  has the structure of a triangulated category. Before describing its translation functor, we first recall a definition and some facts. Given a finite-dimensional  $\mathbb{k}$ -algebra  $A$ , its *Nakayama functor* is given by  $\nu = D \text{Hom}_A(-, A) : \text{mod-}A \rightarrow \text{mod-}A$ , where  $D$  is the duality with the field. If  $A$  is self-injective, then  $\nu$  restricts to an equivalence  $\text{proj-}A \rightarrow \text{proj-}A$ .

If  $Q$  is a Dynkin quiver, then  $\Pi(Q)$  is self-injective and finite-dimensional, and so  $\nu$  gives rise to an autoequivalence of  $\text{proj-}\Pi(Q)$ . In fact, if we identify the indecomposable projective modules  $e_i \Pi(Q)$  with the vertices of  $Q$  in the obvious way, then  $\nu$  induces a graph automorphism.

**Proposition 4.2** ([BES07, Proposition 2.1]). *Let  $Q$  be a Dynkin quiver. Then the Nakayama functor  $\nu$  of  $\Pi(Q)$  induces the identity automorphism of  $Q$  when  $Q$  is  $\mathbb{A}_1, \mathbb{D}_n$  ( $n$  even),  $\mathbb{E}_7$ , or  $\mathbb{E}_8$ , and it is the unique graph automorphism of order 2 when  $Q$  is  $\mathbb{A}_n$  ( $n \geq 2$ ),  $\mathbb{D}_n$  ( $n$  odd), or  $\mathbb{E}_6$ .*

If  $Q$  is Dynkin, we write  $\pi_Q$  for the graph automorphism of  $Q$  induced by  $\nu$ , and we call this the *Nakayama automorphism* of  $Q$ . We also speak of the Nakayama automorphism of a disjoint union of Dynkin graphs, which acts as the Nakayama automorphism on each connected component. The translation functor on  $\text{proj-}\Pi(Q)$  with its canonical triangulated structure is induced by the Nakayama automorphism, in the sense that  $\Sigma e_i \Pi(Q) = e_{\pi_Q(i)} \Pi(Q)$  ([Ami07, Corollary 9.3]). Moreover, by [Ami07, Theorem 7.2], there is a unique triangulated structure on  $\text{proj-}\Pi(Q)$  having this translation functor. Henceforth, whenever we speak of *the* triangulated

category  $\text{proj-}\Pi(Q)$ , we mean  $\text{proj-}\Pi(Q)$  with this triangulated structure.

To determine the triangulated structure of  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ , one must determine the translation functor  $\Sigma$ . The majority of the remainder of this paper will be dedicated to this. The crucial fact is the following, which is established in Sections 6, 9, and 10, and says that  $\Sigma$  induces the Nakayama automorphism on the connected components of  $Q_\lambda$ .

**Theorem 4.3.** *Let  $\tilde{Q}$  be an extended Dynkin quiver, and write  $Q$  for the subquiver obtained by deleting the extending vertex. Let  $\lambda$  be a quasi-dominant weight, and consider the full subquiver  $Q_\lambda$  of  $Q$ . Let  $i \in I_\lambda$ , so  $V_i$  is a nonzero indecomposable object in  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ , and let  $Q'$  be the maximal connected component of  $Q_\lambda$  containing vertex  $i$ , which will necessarily be Dynkin. Then  $\Sigma V_i = V_{\pi_{Q'}(i)}$ .*

Since  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$  is Krull-Schmidt with indecomposable objects given by non-projective vertex modules, this completely determines the translation functor. We postpone the proof of Theorem 4.3 for now, as it requires a case-by-base analysis which the majority of the rest of the paper is devoted to. However, with Theorem 4.3 in hand we are able to prove our main result:

**Theorem 4.4.** *Let  $\tilde{Q}$ ,  $Q$ , and  $\lambda$  be as above, and suppose that  $Q_\lambda = Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$ , where the  $Q^{(i)}$  are connected and necessarily Dynkin. Then the  $\mathbb{k}$ -linear equivalence*

$$\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)}),$$

of Corollary 3.10 is in fact a triangle equivalence, where the translation functor acts on vertex modules as described in Theorem 4.3. In other words, the induced triangulated structure on  $\bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$  agrees with the canonical triangulated structure defined by Amiot.

*Proof.* For each  $1 \leq i \leq r$ , define  $\mathcal{T}_i := \text{add}_{\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda} \left( \bigoplus_{j \in Q_0^{(i)}} V_j \right)$ . We claim that the  $\mathcal{T}_i$  are triangulated subcategories of  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ . Fix  $i$  as above. We need to show that the following hold:

- (1)  $\mathcal{T}_i$  is closed under  $\Sigma$  and  $\Sigma^{-1}$ ; and
- (2) If  $f : X \rightarrow Y$  is a morphism in  $\mathcal{T}_i$  which we extend to a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h} \Sigma X, \tag{4.5}$$

then  $Z$  is isomorphic to an object of  $\mathcal{T}_i$ .

The first of these properties follows from Theorem 4.3, so it remains to show that (2) holds. So let  $f : X \rightarrow Y$  be a morphism, where  $X, Y \in \mathcal{T}_i$ ; in particular,  $X$  and  $Y$  are direct sums of vertex modules where the vertices lie in  $Q_0^{(i)}$ . Extend this to a triangle as in 4.5 where  $Z \in \underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ , and so  $Z$  is a direct sum of vertex modules. Consider the object

$$M = \bigoplus_{j \in (Q_\lambda)_0 \setminus (Q^{(i)})_0} V_j.$$

The functor  $\underline{\text{Hom}}_{\mathcal{O}^\lambda}(M, -)$  is homological, so in particular there is an exact sequence

$$\underline{\text{Hom}}_{\mathcal{O}^\lambda}(M, Y) \rightarrow \underline{\text{Hom}}_{\mathcal{O}^\lambda}(M, Z) \rightarrow \underline{\text{Hom}}_{\mathcal{O}^\lambda}(M, \Sigma X).$$

The flanking terms are both 0 because the only morphism between two vertex modules in different connected components of  $Q_\lambda$  is the zero morphism. Therefore also  $\underline{\text{Hom}}_{\mathcal{O}^\lambda}(M, Z) = 0$ , which implies that every direct summand of  $Z$  must lie in  $\mathcal{T}_i$ , and so (2) holds.

It follows that we have a triangle equivalence

$$\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^r \mathcal{T}_i.$$

It remains to show that, for each  $i$ ,  $\mathcal{T}_i$  is triangle equivalent to  $\text{proj-}\Pi(Q^{(i)})$ ; the proof of this is similar to the proof of Theorem 3.6. Write  $e^{(i)} = \sum_{j \in Q_0^{(i)}} e_j$ , observing that  $e^{(i)} \Pi^\lambda e_0 = \bigoplus_{j \in Q_0^{(i)}} V_j$ , and consider the functor

$$\underline{\text{Hom}}_{\mathcal{O}^\lambda}(e^{(i)} \Pi^\lambda e_0, -) : \underline{\text{mod-}}\mathcal{O}^\lambda \rightarrow \text{mod-}\underline{\text{End}}_{\mathcal{O}^\lambda}(e^{(i)} \Pi^\lambda e_0).$$

As in the proof of Theorem 3.6, this induces a fully faithful functor  $\mathcal{T}_i = \text{add}\left(e^{(i)} \Pi^\lambda e_0\right) \rightarrow \text{proj-}\Pi(Q^{(i)})$ , which, as in the proof of Theorem 3.6, is actually a  $\mathbb{k}$ -linear equivalence  $\mathcal{T}_i \rightarrow \text{proj-}\Pi(Q^{(i)})$ . Now each  $\mathcal{T}_i$  is triangulated with translation functor given by the Nakayama automorphism, by Theorem 4.3, and this induces a triangulated structure on  $\text{proj-}\Pi(Q^{(i)})$ . Since the triangulated structure on  $\text{proj-}\Pi(Q^{(i)})$  with this translation functor is unique ([Ami07, Theorem 7.2]) we have a triangle equivalence between  $\mathcal{T}_i$  and  $\text{proj-}\Pi(Q^{(i)})$ .  $\square$

In particular, we can now prove Theorem 1.2 from the introduction, where we recall that we write  $R_Q$  for the coordinate ring of a Kleinian singularity with Dynkin quiver  $Q$ :

**Theorem 4.6.** *Let  $\tilde{Q}$  and  $Q$  be as above, and let  $\lambda \in \mathbb{k}^{n+1}$  be a weight for  $\tilde{Q}$ . Then there exists a subset  $J$  of  $\{1, \dots, n\}$  such that, if  $Q^{(1)} \sqcup \dots \sqcup Q^{(r)}$  is the full subquiver of  $Q$  obtained by deleting the vertices in  $J$ , so that the  $Q^{(i)}$  are connected and therefore necessarily Dynkin, there is a triangle equivalence*

$$\underline{\text{MCM-}}\mathcal{O}^\lambda(\tilde{Q}) \simeq \bigoplus_{i=1}^r \underline{\text{MCM-}}R_{Q^{(i)}}.$$

*Proof.* By Lemma 2.14 there is a quasi-dominant weight  $\lambda'$  with  $\mathcal{O}^\lambda \cong \mathcal{O}^{\lambda'}$ . Writing  $J = \{i \in \{1, \dots, n\} \mid \lambda'_i = 0\}$ , Theorem 4.4 tells us that there are triangle equivalences

$$\underline{\text{MCM-}}\mathcal{O}^\lambda \simeq \underline{\text{MCM-}}\mathcal{O}^{\lambda'} \simeq \bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)}).$$

But Theorem 4.4 also tells us that there are triangle equivalences  $\underline{\text{MCM-}}R_{Q^{(i)}} \simeq \text{proj-}\Pi(Q^{(i)})$ , whence the result.  $\square$

**Corollary 4.7.** *Let  $\tilde{Q}$  and  $\tilde{Q}'$  be extended Dynkin quivers (not necessarily the same type) and let  $\lambda$  and  $\lambda'$  be quasi-dominant weights for  $\tilde{Q}$  and  $\tilde{Q}'$ , respectively. Suppose that  $Q_\lambda$  and  $Q_{\lambda'}$  are equal to the same disjoint union of Dynkin quivers,  $Q^{(1)} \sqcup \dots \sqcup Q^{(r)}$ . Then there is a triangle equivalence*

$$\underline{\text{MCM-}}\mathcal{O}^\lambda(\tilde{Q}) \simeq \underline{\text{MCM-}}\mathcal{O}^{\lambda'}(\tilde{Q}').$$

*Remark 4.8.* As with Corollary 3.11, this triangle equivalence fails to be unique. For example, the  $\mathbb{k}$ -linear equivalences in Example 3.13 are both triangle equivalences.

It remains to prove Theorem 4.3. When  $\tilde{Q} = \tilde{A}_n$ , the structure of  $\mathcal{O}^\lambda$  is well-understood, and so we can use ring theoretic techniques similar to those found in [Hod93] to determine the translation functor. On the other hand, if  $\tilde{Q}$  is of type  $\tilde{D}$  or  $\tilde{E}$  then calculations in  $\mathcal{O}^\lambda$  are much more difficult, especially when it is noncommutative. In these cases, we instead make use of some commutative ring-theoretic results, and explain how they are useful for our problem. Unfortunately, this requires a case-by-case argument; ideally, one would be able to prove Theorem 4.3 without having to resort to this.

Before proving Theorem 4.3, we turn our attention to a noncommutative geometric McKay correspondence and prove Theorem 1.4.



## 5. A NONCOMMUTATIVE GEOMETRIC MCKAY CORRESPONDENCE

Let  $\tilde{Q}$  be an extended Dynkin quiver with  $n + 1$  vertices, let  $Q$  be the quiver obtained by removing the extending vertex, and let  $\lambda = \varepsilon_0 = (1, 0, \dots, 0)$ ; that is, the weight at the extending vertex is 1, and 0 for all of the other vertices. We may then consider  $\mathcal{O}^\lambda(\tilde{Q})$  to be a noncommutative analogue of  $R = \mathcal{O}(\tilde{Q})$ , the coordinate ring of the corresponding Kleinian singularity; indeed, we have just seen that  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \underline{\text{MCM}}\text{-}R$ . We now provide another reason why  $\mathcal{O}^\lambda$  may be considered a noncommutative analogue of  $R$ .

Inspired, for example, by [VdB04], we say that a  $\mathbb{k}$ -algebra  $S$  is a *noncommutative resolution* of a  $\mathbb{k}$ -algebra  $R$  if  $S = \text{End}_R(M)$  for some generator  $M$  and  $\text{gl.dim } S < \infty$ . Moreover, motivated by [MS01], given  $M, N \in \text{mod-}S$  which satisfy  $\dim_{\mathbb{k}} \text{Ext}_S^\ell(M, N) < \infty$  for all  $\ell \geq 0$ , we define the *intersection multiplicity* of  $M$  and  $N$  to be

$$M \bullet N := \sum_{\ell \geq 0} (-1)^{\ell+1} \dim_{\mathbb{k}} \text{Ext}_S^\ell(M, N)$$

(note that this sum has finitely many terms since  $S$  is smooth).

**5.1. Intersection theory for a family of noncommutative resolutions.** We return now to the  $\mathbb{k}$ -algebra of interest, namely  $\mathcal{O}^\lambda$  where  $\lambda = \varepsilon_0$ . Our first aim is to identify an appropriate noncommutative resolution, which we have in fact already done:

**Lemma 5.1.**  $\Pi^\lambda$  is a noncommutative resolution of  $\mathcal{O}^\lambda$ .

*Proof.* First note that the  $\mathcal{O}^\lambda$ -module  $\Pi^\lambda e_0$  has  $e_0 \Pi^\lambda e_0 = \mathcal{O}^\lambda$  as a direct summand, and so is a generator. Moreover,  $\Pi^\lambda = \text{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0)$  by Lemma 3.1, and this is smooth by [CBH98, Theorem 1.5]. Therefore  $\Pi^\lambda$  is a noncommutative resolution of  $\mathcal{O}^\lambda$  by definition.  $\square$

We can actually obtain infinitely many noncommutative resolutions of  $\mathcal{O}^\lambda$  using the so-called *dual reflections* of [CBH98], whose definition we now recall.

Suppose that  $\tilde{Q}$  is an extended Dynkin graph with  $(n + 1) \times (n + 1)$  (generalised) Cartan matrix  $\tilde{C}$ , i.e.,  $\tilde{C} = 2I - A$ , where  $A$  is the adjacency matrix of the underlying graph of  $\tilde{Q}$ . We index the rows and columns of  $\tilde{C}$  by the integers  $0, 1, \dots, n$ . For each vertex  $i \in \tilde{Q}_0$ , define a *dual reflection*  $r_i : \mathbb{k}^{\tilde{Q}_0} \rightarrow \mathbb{k}^{\tilde{Q}_0}$  by

$$(r_i \lambda)_j = \lambda_j - \tilde{C}_{ij} \lambda_i.$$

It is clear that the  $r_i$  preserve the  $\mathbb{Z}^{n+1}$  lattice inside  $\mathbb{k}^{n+1}$ . Moreover, it is not difficult to show that  $\lambda \cdot \delta = r_i \lambda \cdot \delta$  for all  $\lambda \in \mathbb{k}^{\tilde{Q}_0}$  and  $i \in \tilde{Q}_0$ , so that the  $r_i$  preserve the affine hyperplanes  $\{\lambda \in \mathbb{k}^{n+1} \mid \lambda \cdot \delta = c\}$  for each  $c \in \mathbb{k}$ ; since  $\varepsilon_0 \cdot \delta = 1$ , we are primarily interested in the case  $c = 1$ . Then we have the following:

**Theorem 5.2** ([CBH98, Corollary 5.2]). *Let  $\rho$  be a composition of dual reflections. Then  $\Pi^\lambda$  is Morita equivalent to  $\Pi^{\rho(\lambda)}$ .*

By combining Lemma 5.1 and Theorem 5.2, we obtain the following:

**Corollary 5.3.**  $\Pi^{\rho(\lambda)}$  is a noncommutative resolution of  $\mathcal{O}^\lambda$  for any composition of dual reflections  $\rho$ .

To establish a noncommutative version of the geometric McKay correspondence, we first identify an analogue of the exceptional lines appearing in a resolution of a Kleinian singularity. When  $\lambda = \varepsilon_0$ , by [CBH98, Lemma 7.1 (2)],  $\Pi^\lambda$  has precisely  $n$  isoclasses of finite-dimensional simple modules, and hence by Morita equivalence so too does  $\mathcal{O}^{\rho(\lambda)}$  for any composition of dual reflections  $\rho$ . These will play the role of the exceptional objects in our noncommutative

resolution. Before we can prove a result justifying this, we have the following lemma, the proof of which is deferred to Sections 6, 9, and 10:

**Lemma 5.4.** *Let  $\tilde{Q}$  be an extended Dynkin quiver and  $\lambda = \varepsilon_0$ . Fix a vertex  $0 \neq i \in \tilde{Q}_0$  and write  $\partial i$  for the set of vertices adjacent to  $i$  in  $\tilde{Q}$ . Then there is a short exact sequence of  $\mathcal{O}^\lambda$ -modules*

$$0 \rightarrow V_i \rightarrow \bigoplus_{k \in \partial i} V_k \rightarrow V_i \rightarrow 0. \quad (5.5)$$

This allows us to prove a preliminary version of Theorem 1.4 from the introduction:

**Theorem 5.6.** *Let  $\tilde{Q}$  be an extended Dynkin quiver with  $n+1$  vertices, and let  $\lambda = \varepsilon_0$ . Let  $\mu = \rho(\lambda)$ , where  $\rho$  is any composition of dual reflections, so that  $\Pi^\mu$  is a noncommutative resolution of  $\mathcal{O}^\lambda$ . Then  $\Pi^\mu$  has precisely  $n$  finite-dimensional simple modules  $S_i$  up to isomorphism, and with a suitable indexing of them, the intersection matrix  $\Gamma$  with entries  $\Gamma_{ij} = S_i \bullet S_j$  is equal to  $(-1)$  times  $C$ , where  $C$  is the Cartan matrix of Dynkin type corresponding to  $Q$ .*

*Proof.* The discussion preceding Lemma 5.4 shows that  $\Pi^\mu$  has  $n$  finite-dimensional simple modules  $S_i$  up to isomorphism, so it remains to prove the result on the intersection multiplicities.

Since Morita equivalence preserves dimensions of Hom and Ext groups, we are able to calculate the intersection numbers of the finite-dimensional  $\Pi^\mu$ -modules by doing the calculations in  $\Pi^\lambda$  instead. Identifying  $\Pi^\lambda$ -modules with representations of  $\tilde{Q}$  which satisfy the relations coming from  $\Pi^\lambda$ , [CBH98, Lemma 7.2 (6), Theorem 7.4] tells us that the dimension vector of  $S_i$  is  $\varepsilon_i \in \mathbb{N}^{n+1}$ . It follows that

$$S_i \cong \frac{e_i \Pi^\lambda}{\bigoplus_{\substack{\alpha \in \tilde{Q}_1 \\ t(\alpha)=i}} \alpha \Pi^\lambda} \quad (5.7)$$

is a representative for  $S_i$ . Also observe that

$$\mathrm{Hom}_{\Pi^\lambda}(e_i \Pi^\lambda, S_j) = \begin{cases} \mathbb{k} e_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \quad (5.8)$$

By Lemma 5.4, for  $1 \leq i \leq n$  we have a short exact sequence of the form (5.5). In this sequence, a morphism between two vertex modules is given by left multiplication by the arrow between them in the double of  $\tilde{Q}$ . Applying the functor  $\mathrm{Hom}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -)$  and using Corollary 3.2, we obtain an exact sequence

$$0 \rightarrow e_i \Pi^\lambda \rightarrow \bigoplus_{k \in \partial i} e_k \Pi^\lambda \rightarrow e_i \Pi^\lambda$$

where again the morphism between two vertex modules is given by the arrow between them. By (5.7), the cokernel of the right hand map is  $S_i$  and, noting that the modules  $e_k \Pi^\lambda$  are direct summands of  $\Pi^\lambda$  and hence projective, we have the following projective resolution of  $S_i$ :

$$0 \rightarrow e_i \Pi^\lambda \rightarrow \bigoplus_{k \in \partial i} e_k \Pi^\lambda \rightarrow e_i \Pi^\lambda \rightarrow S_i \rightarrow 0.$$

Now let  $1 \leq j \leq n$ . Seeking to calculate the extension groups between  $S_i$  and  $S_j$ , we apply  $\mathrm{Hom}_{\Pi^\lambda}(-, S_j)$  to the corresponding deleted resolution to obtain the complex

$$0 \rightarrow \mathrm{Hom}_{\Pi^\lambda}(e_i \Pi^\lambda, S_j) \rightarrow \bigoplus_{k \in \partial i} \mathrm{Hom}_{\Pi^\lambda}(e_k \Pi^\lambda, S_j) \rightarrow \mathrm{Hom}_{\Pi^\lambda}(e_i \Pi^\lambda, S_j) \rightarrow 0. \quad (5.9)$$

We now consider three distinct cases when computing the homology of this complex. If  $j = i$  then, using (5.8), as a complex of vector spaces (5.9) becomes

$$0 \rightarrow \mathbb{k} \rightarrow 0 \rightarrow \mathbb{k} \rightarrow 0$$

and so we can immediately read off that

$$\dim_{\mathbb{k}} \operatorname{Hom}_{\Pi^\lambda}(S_i, S_i) = 1 = \dim_{\mathbb{k}} \operatorname{Ext}_{\Pi^\lambda}^2(S_i, S_i), \quad \dim_{\mathbb{k}} \operatorname{Ext}_{\Pi^\lambda}^\ell(S_i, S_i) = 0 \quad \text{for } \ell = 1 \text{ or } \ell \geq 3,$$

and so  $S_i \bullet S_i = -1 + 0 - 1 = -2$ . If  $j \in \partial i$ , then (5.9) becomes

$$0 \rightarrow 0 \rightarrow \mathbb{k} \rightarrow 0 \rightarrow 0$$

and so

$$\dim_{\mathbb{k}} \operatorname{Ext}_{\Pi^\lambda}^1(S_i, S_i) = 1, \quad \dim_{\mathbb{k}} \operatorname{Ext}_{\Pi^\lambda}^\ell(S_i, S_i) = 0 \quad \text{for } \ell = 0 \text{ or } \ell \geq 2.$$

That is, if  $i$  and  $j$  are adjacent in  $\tilde{Q}$ , then  $S_i \bullet S_j = 0 + 1 + 0 = 1$ . Finally, if  $j \neq i$  and  $j \notin \partial i$  then (5.9) becomes

$$0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0$$

and clearly

$$\dim_{\mathbb{k}} \operatorname{Ext}_{\Pi^\lambda}^\ell(S_i, S_i) = 0 \quad \text{for } \ell \geq 0,$$

and so  $S_i \bullet S_j = 0$  in this case. It follows that the intersection matrix  $\Gamma$  satisfies  $\Gamma = -C$ .  $\square$

The above result should be seen as a noncommutative analogue of the geometric McKay correspondence. However, we can strengthen this result by showing that  $\mathcal{O}^\lambda$  possesses a noncommutative resolution which is actually a “deformation”: that is, a noncommutative resolution of the form  $\mathcal{O}^\mu$  for some weight  $\mu$ . Since we are restricting our attention to quasi-dominant weights, smoothness of  $\mathcal{O}^\mu$  forces  $\mu_i \succ 0$  for all  $i \geq 1$  (c.f. Lemma 2.15). It is not immediately clear that such a deformation exists; we prove its existence in the next subsection.

**5.2.  $\mathcal{O}^\lambda$  has a noncommutative resolution which is a deformation.** The dual reflections mentioned previously also appear in the so-called *numbers game* of [Moz90]. The relationship between this game and our setting is that the moves considered by Mozes can equivalently be described as an application of a dual reflection to a weight  $\lambda$ . This allows us to make use of some of the results from this paper; in particular we are able to prove that, for  $\lambda = \varepsilon_0$ , noncommutative resolutions of  $\mathcal{O}^\lambda$  which are also deformations exist:

**Lemma 5.10.** *Let  $\tilde{Q}$  be an extended Dynkin quiver with  $n + 1$  vertices, and let  $\lambda = \varepsilon_0$ . Then there exists a sequence of dual reflections  $\rho$  such that  $\rho(\lambda)_i > 0$  for all  $i \neq 0$ ; in particular,  $\rho(\lambda)$  is quasi-dominant.*

*Proof.* Write  $G$  for the group generated by the dual reflections. Lemma 5.5 of [Moz90], when translated into our notation, says that  $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda_i \geq 0 \text{ for all } 0 \leq i \leq n\}$  is a fundamental domain for the action of  $G$  on  $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda \cdot \delta > 0\}$ . Recalling that  $G$  preserves the affine hyperplane  $V := \{\lambda \in \mathbb{R}^{n+1} \mid \lambda \cdot \delta = 1\}$ , it follows that  $V = \bigcup_{\rho \in G} \rho U$ , where  $U$  is the  $n$ -simplex  $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda_i \geq 0 \text{ for all } 0 \leq i \leq n \text{ and } \lambda \cdot \delta = 1\}$ . Let  $H = \{\lambda \in V \mid \lambda_i > 0 \text{ for all } i \neq 0\}$ , which is a convex subset of  $V$  containing open balls of arbitrarily large diameter. Since each  $\rho U$  has the same finite diameter, there exists some  $\rho \in G$  with  $\rho U \subseteq H$ . In particular,  $\rho(\lambda) \in H$ ; that is,  $\rho(\lambda)_i > 0$  for all  $i \neq 0$ .  $\square$

*Remark 5.11.* By playing Mozes’ numbers game, one can often determine an explicit sequence of dual reflections  $\rho$  satisfying the hypotheses of Lemma 5.10. For example, if  $\tilde{Q} = \mathbb{A}_4$ , then the numbers game starting with the initial configuration  $(-3, 1, 1, 1)$  terminates at  $\varepsilon_0$ , and so by applying the corresponding dual reflections in reverse we obtain the desired  $\rho$ . More generally, [GSS12, Proposition 5.1] tells us that when  $\tilde{Q}$  is of type  $\tilde{\mathbb{A}}_{2m}$ ,  $\tilde{\mathbb{D}}_{4m}$ ,  $\tilde{\mathbb{D}}_{4m+1}$ ,  $\tilde{\mathbb{E}}_6$  or  $\tilde{\mathbb{E}}_8$ , where  $m$  is a positive integer, then the numbers game starting with the initial configuration  $(1 - \sum_{i=1}^n \delta_i, 1, 1, \dots, 1)$  terminates at  $\varepsilon_0$ , and so this determines a sequence of dual reflections  $\rho$  such that  $\rho(\lambda)_i > 0$  for all  $i \neq 0$ .

We are now in a position to prove Theorem 1.4 from the introduction:

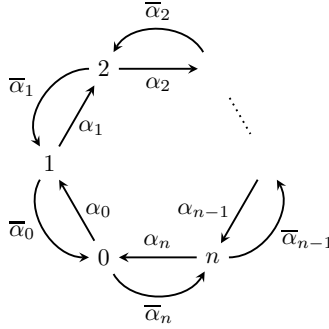
**Theorem 5.12.** *Let  $\tilde{Q}$  be an extended Dynkin quiver with  $n + 1$  vertices, and let  $\lambda = \varepsilon_0$ . Then  $\mathcal{O}^\lambda$  has a noncommutative resolution of the form  $\mathcal{O}^\mu$ , where  $\mathcal{O}^\mu$  has precisely  $n$  finite-dimensional simple modules  $S_i$  up to isomorphism. With a suitable indexing of the  $S_i$ , the intersection matrix  $\Gamma$  with entries  $\Gamma_{ij} = S_i \bullet S_j$  is equal to  $(-1)$  times  $C$ , where  $C$  is the Cartan matrix of Dynkin type corresponding to  $Q$ .*

*Proof.* Lemma 5.10 tells us that there exists a sequence of dual reflections  $\rho$  such that  $\mathcal{O}^\mu$  is smooth, where  $\mu = \rho(\lambda)$ . Since  $\Pi^\lambda$  is a resolution of  $\mathcal{O}^\lambda$  and there are Morita equivalences between  $\Pi^\lambda$ ,  $\Pi^\mu$ , and  $\mathcal{O}^\mu$  (by [CBH98, Corollary 5.2, Corollary 9.6]), it follows that  $\mathcal{O}^\mu$  is a noncommutative resolution of  $\mathcal{O}^\lambda$ . Finally, these Morita equivalences combined with Theorem 5.6 tells us that  $\mathcal{O}^\mu$  has precisely  $n$  finite-dimensional simple modules  $S_i$  up to isomorphism, and since Morita equivalences preserve dimensions of Hom and Ext groups, the claimed intersection multiplicities follow from Theorem 5.6 as well.  $\square$

We now return to the proof of Theorem 4.3, which comprises the remainder of the paper.

## 6. THE TRANSLATION FUNCTOR IN TYPE $\mathbb{A}$

Let  $\tilde{Q}$  be an extended Dynkin quiver of type  $\tilde{\mathbb{A}}_n$ , and fix an orientation of its arrows so that its underlying double quiver is as in Figure 1, which we recall below.



The relations in  $\Pi^\lambda$  are given by  $\alpha_i \bar{\alpha}_i - \bar{\alpha}_{i-1} \alpha_{i-1} = \lambda_i e_i$ , where the subscripts are read modulo  $n + 1$ . In later calculations, we will often omit the vertex idempotent in this equality when there is no possibility for confusion.

In this case,  $\mathcal{O}^\lambda$  is generated as a  $\mathbb{k}$ -algebra by the elements

$$x = \alpha_0 \alpha_1 \dots \alpha_n, \quad y = \bar{\alpha}_n \bar{\alpha}_{n-1} \dots \bar{\alpha}_0, \quad z = \alpha_0 \bar{\alpha}_0,$$

and it can be checked that  $\mathcal{O}^\lambda$  has a presentation of the form

$$\overline{\mathbb{k}\langle x, y, z \rangle}, \quad \left\langle \begin{array}{ll} xz = (z + \lambda \cdot \delta)x, & xy = \prod_{i=0}^n \left( z + \sum_{j=1}^i \lambda_j \right) \\ yz = (z - \lambda \cdot \delta)y, & yx = \prod_{i=0}^n \left( z - \lambda \cdot \delta + \sum_{j=1}^i \lambda_j \right) \end{array} \right\rangle,$$

where we remind the reader that  $\delta = (1, 1, \dots, 1)$ . Since there is no loss in generality assuming  $\lambda \cdot \delta = 1$ , these are precisely the algebras considered by Hodges in [Hod93].

We define a grading on  $\Pi^\lambda$  by declaring the degree of each  $\alpha_i$  to be 1, the degree of each  $\bar{\alpha}_i$  to be  $-1$ , and the degree of each vertex idempotent  $e_i$  to be 0. Since the relations are homogeneous, this induces a  $\mathbb{Z}$ -grading on  $\mathcal{O}^\lambda$  (note that  $\mathcal{O}_m^\lambda$  is nonzero precisely when  $m$  is an integer multiple

of  $n + 1$ ). We then observe that the graded decomposition of  $\mathcal{O}^\lambda$  is given by a decomposition into a direct sum of distinct  $\alpha_0 \bar{\alpha}_0$ -weight spaces:

$$\mathcal{O}^\lambda = \bigoplus_{m=0}^{\infty} (\alpha_0 \dots \alpha_n)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0] \oplus \bigoplus_{m=1}^{\infty} (\bar{\alpha}_n \dots \bar{\alpha}_0)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0]. \quad (6.1)$$

Using the fact that  $(\alpha_0 \dots \alpha_n)^m \alpha_0 \bar{\alpha}_0 = q(\alpha_0 \bar{\alpha}_0)(\alpha_0 \dots \alpha_n)^m$  for some polynomial  $q$ , and similarly  $(\bar{\alpha}_n \dots \bar{\alpha}_0)^m \alpha_0 \bar{\alpha}_0 = q'(\alpha_0 \bar{\alpha}_0)(\bar{\alpha}_n \dots \bar{\alpha}_0)^m$  for some  $q'$ , it follows that we also have a decomposition

$$\mathcal{O}^\lambda = \bigoplus_{m=0}^{\infty} \mathbb{k}[\alpha_0 \bar{\alpha}_0](\alpha_0 \dots \alpha_n)^m \oplus \bigoplus_{m=1}^{\infty} \mathbb{k}[\alpha_0 \bar{\alpha}_0](\bar{\alpha}_n \dots \bar{\alpha}_0)^m.$$

In the remainder of this section, we will frequently make use of the following identities which the reader is encouraged to check: if  $0 \leq i \leq j \leq n$ , then

$$\alpha_i \bar{\alpha}_i \bar{\alpha}_{i-1} \dots \bar{\alpha}_1 \bar{\alpha}_0 = \bar{\alpha}_{i-1} \bar{\alpha}_{i-2} \dots \bar{\alpha}_1 \bar{\alpha}_i (\alpha_0 \bar{\alpha}_0 + \lambda_1 + \dots + \lambda_i), \quad (6.2)$$

$$\alpha_i \dots \alpha_j \bar{\alpha}_j \dots \bar{\alpha}_i = \prod_{k=i}^j \left( \bar{\alpha}_{i-1} \alpha_{i-1} + \sum_{\ell=k}^j \lambda_{\ell-1} \right). \quad (6.3)$$

**Lemma 6.4.**

(1) If  $1 \leq i \leq n$ , then

$$V_i = \bar{\alpha}_{i-1} \dots \bar{\alpha}_0 \mathcal{O}^\lambda + \alpha_i \dots \alpha_n \mathcal{O}^\lambda.$$

(2) For any  $0 \leq i \leq n + 1$ , we have

$$V_i = \bigoplus_{m=0}^{\infty} \alpha_i \dots \alpha_n (\alpha_0 \dots \alpha_n)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0] \oplus \bigoplus_{m=0}^{\infty} \bar{\alpha}_{i-1} \dots \bar{\alpha}_0 (\bar{\alpha}_n \dots \bar{\alpha}_0)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0].$$

*Remark 6.5.* It will become clear later on why we allow for  $i = 0$  and  $i = n + 1$  in part (2) of the above lemma. In these cases, the claimed result is consistent with (6.1) if one interprets  $\bar{\alpha}_{i-1} \dots \bar{\alpha}_0$  or  $\alpha_i \dots \alpha_n$ , respectively, as an empty product.

*Proof of Lemma 6.4.*

(1) We first prove the following statement by induction on  $r \geq 0$ : if  $p$  is a nonzero path in  $\bar{\mathcal{Q}}$  starting at vertex  $j$ , where  $1 \leq j \leq n$ , of length  $j + r$  which ends with  $\bar{\alpha}_0$  and which visits vertex 0 only at the end of the path, then we can use the relations in  $\mathcal{O}^\lambda$  to write  $p = \bar{\alpha}_{j-1} \dots \bar{\alpha}_0 q$ , where  $q \in \mathcal{O}^\lambda$ .

The case  $r = 0$  is clear, as the only path of length  $j$  of the required form is  $\bar{\alpha}_{j-1} \dots \bar{\alpha}_0$ . So suppose that  $p$  is a path starting at vertex  $j$  which has length  $j + r$ , where  $r \geq 1$ , and which has the required form. If the first arrow in  $p$  is  $\alpha_j$ , then  $p = \alpha_j p'$ , where  $p'$  starts at vertex  $j + 1$  and has length  $j + r - 1 = (j + 1) + (r - 2)$ . By induction,  $p' = \bar{\alpha}_j \bar{\alpha}_{j-1} \dots \bar{\alpha}_0 q$ , where  $q \in \mathcal{O}^\lambda$ . Then

$$p = \alpha_j \bar{\alpha}_j \bar{\alpha}_{j-1} \dots \bar{\alpha}_0 q = \bar{\alpha}_{j-1} \dots \bar{\alpha}_2 \bar{\alpha}_1 \bar{\alpha}_0 \underbrace{(\alpha_0 \bar{\alpha}_0 + \lambda_1 + \lambda_2 + \dots + \lambda_j)}_{\in \mathcal{O}^\lambda} q,$$

where we make use of (6.2), and so  $p$  has the claimed form.

If instead the first arrow in  $p$  is  $\bar{\alpha}_{j-1}$ , then write  $p = \bar{\alpha}_{j-1} p'$ . Then if the first arrow of  $p'$  is  $\alpha_{j-1}$ , the same argument as above shows that  $p' \in \bar{\alpha}_{j-2} \dots \bar{\alpha}_0 \mathcal{O}^\lambda$ , whence  $p$  has the claimed form. Alternatively, if the first arrow of  $p'$  is  $\bar{\alpha}_{j-2}$ , then write  $p' = \bar{\alpha}_{j-2} p''$ , so that  $p = \bar{\alpha}_{j-1} \bar{\alpha}_{j-2} p''$ , and run the same argument again. This procedure terminates in finitely many steps, and shows that  $p$  has the required form, proving the claim.

The lemma now follows quickly: consider any path  $p$  starting at vertex  $i$ , which we may as well assume goes through vertex 0 only once, and that this occurs at the end of the path. If the

final arrow in  $p$  is  $\bar{\alpha}_0$ , then the above result shows that  $p \in \bar{\alpha}_{i-1} \dots \bar{\alpha}_0 \mathcal{O}^\lambda$ , while if the final arrow is  $\alpha_n$ , a symmetrical argument shows that  $p \in \alpha_i \dots \alpha_n \mathcal{O}^\lambda$ .

(2) We seek to make use of part (1) and (6.1). For  $m \geq 1$ , we have

$$\begin{aligned} \bar{\alpha}_{i-1} \dots \bar{\alpha}_0 (\alpha_0 \dots \alpha_n)^m &= \bar{\alpha}_{i-1} \dots \bar{\alpha}_0 \alpha_0 \dots \alpha_{i-1} \alpha_i \dots \alpha_n (\alpha_0 \dots \alpha_n)^{m-1} \\ &= p(\alpha_i \bar{\alpha}_i) \alpha_i \dots \alpha_n (\alpha_0 \dots \alpha_n)^{m-1}, \end{aligned}$$

for some polynomial  $p$ , by using the defining relations of  $\Pi^\lambda$ . This is then equal to

$$\begin{aligned} \alpha_i p(\bar{\alpha}_i \alpha_i) \alpha_{i+1} \dots \alpha_n (\alpha_0 \dots \alpha_n)^{m-1} &= \alpha_i p(\alpha_{i+1} \bar{\alpha}_{i+1} - \lambda_{i+1}) \alpha_{i+1} \dots \alpha_n (\alpha_0 \dots \alpha_n)^{m-1} \\ &\vdots \\ &= \alpha_i \dots \alpha_n p'(\alpha_0 \bar{\alpha}_0) (\alpha_0 \dots \alpha_n)^{m-1}, \end{aligned}$$

where  $p'(\alpha_0 \bar{\alpha}_0) = p(\alpha_0 \bar{\alpha}_0 - (\lambda_{i+1} + \dots + \lambda_n))$ . By a similar argument, one can then move  $(\alpha_0 \dots \alpha_n)^{m-1}$  through  $p'$  to obtain

$$\bar{\alpha}_{i-1} \dots \bar{\alpha}_0 (\alpha_0 \dots \alpha_n)^m = \alpha_i \dots \alpha_n (\alpha_0 \dots \alpha_n)^{m-1} p''(\alpha_0 \bar{\alpha}_0) \quad (6.6)$$

for some polynomial  $p''$ . Similarly, for  $m \geq 1$ ,

$$\alpha_i \dots \alpha_n (\bar{\alpha}_n \dots \bar{\alpha}_0)^m = \bar{\alpha}_{i-1} \dots \bar{\alpha}_0 (\bar{\alpha}_n \dots \bar{\alpha}_0)^{m-1} q(\alpha_0 \bar{\alpha}_0) \quad (6.7)$$

for some polynomial  $q$ . Then, by combining the formula of part (1) with the identities (6.1), (6.6), and (6.7), the claimed decomposition follows.  $\square$

The following proposition allows us to identify a number of short exact sequences of  $\mathcal{O}^\lambda$ -modules; in particular, it proves Lemma 5.4 in the Type  $\mathbb{A}$  case.

**Proposition 6.8.** *Let  $0 \leq i < j \leq n+1$ , and suppose that  $\lambda_m = 0$  for all  $i < m < j$ . Then, for any  $k \geq i$  and  $\ell \leq j$  with  $i+j = k+\ell$ , there is a short exact sequence*

$$0 \rightarrow V_k \rightarrow V_i \oplus V_j \rightarrow V_\ell \rightarrow 0,$$

where if  $j = n+1$  we interpret  $V_{n+1}$  as  $V_0$ .

*Proof.* The case  $i = k, j = \ell$  is clear, so suppose that  $k > j$  and  $\ell < i$ . Define a map

$$\psi : V_i \oplus V_j \rightarrow V_\ell, \quad \psi(p, q) = \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_i p + \alpha_\ell \dots \alpha_{j-1} q.$$

By Lemma 6.4, this map is surjective. We also have that

$$\ker \psi \cong \{q \in V_j \mid \alpha_\ell \dots \alpha_{j-1} q \in \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_i V_i\} \cong \ker(\theta : V_j \rightarrow (V_\ell / \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_i V_i)),$$

where  $\theta$  is given by left multiplication by  $\alpha_\ell \dots \alpha_{j-1}$ . We claim that

$$\alpha_\ell \dots \alpha_{j-1} \cdot \bar{\alpha}_{j-1} \dots \bar{\alpha}_k = \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_i \alpha_i \dots \alpha_{k-1},$$

and so  $\bar{\alpha}_{j-1} \dots \bar{\alpha}_k V_k \subseteq \ker \theta$ . For this, we additionally assume that  $k \geq \ell$ , with the proof when  $k < \ell$  being similar. Indeed, noting that  $k-j = i-\ell$ , and using (6.3) and the hypotheses on the weights,

$$\begin{aligned} \alpha_\ell \dots \alpha_{j-1} \cdot \bar{\alpha}_{j-1} \dots \bar{\alpha}_k &= \alpha_\ell \dots \alpha_{k-1} \alpha_k \dots \alpha_{j-1} \bar{\alpha}_{j-1} \dots \bar{\alpha}_k \\ &= \alpha_\ell \dots \alpha_{k-1} (\bar{\alpha}_{k-1} \alpha_{k-1})^{k-j} \\ &= (\bar{\alpha}_{\ell-1} \alpha_{\ell-1})^{i-\ell} \alpha_\ell \dots \alpha_{k-1} \\ &= \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_i \alpha_i \dots \alpha_{\ell-1} \alpha_\ell \dots \alpha_{k-1} \\ &= \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_i \alpha_i \dots \alpha_{k-1}. \end{aligned}$$

Therefore, noting that  $\bar{\alpha}_{j-1} \dots \bar{\alpha}_k V_k \cong V_k$ , the result follows if we can show that the inclusion  $\bar{\alpha}_{j-1} \dots \bar{\alpha}_k V_k \subseteq \ker \theta$  is an equality. It suffices to prove that the induced map  $V_j / \bar{\alpha}_{j-1} \dots \bar{\alpha}_k V_k \rightarrow$

$V_\ell/\bar{\alpha}_{\ell-1}\dots\bar{\alpha}_iV_i$  is an isomorphism of vector spaces.

With the grading on  $\mathcal{O}^\lambda$  given before Lemma 6.4, we have that (the map induced by)  $\theta$  is a graded map of degree  $j - \ell$ . Since the map induced by  $\theta$  is surjective, it suffices to show that  $V_j/\bar{\alpha}_{j-1}\dots\bar{\alpha}_kV_k$  and  $(V_\ell/\bar{\alpha}_{\ell-1}\dots\bar{\alpha}_iV_i)[j - \ell]$  have the same dimension in each graded piece. We have that

$$\begin{aligned} V_j &= \bigoplus_{m=0}^{\infty} \alpha_j \dots \alpha_n (\alpha_0 \dots \alpha_n)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0] \oplus \bigoplus_{m=0}^{\infty} \bar{\alpha}_{j-1} \dots \bar{\alpha}_0 (\bar{\alpha}_n \dots \bar{\alpha}_0)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0] \\ \bar{\alpha}_{j-1} \dots \bar{\alpha}_k V_k &= \bigoplus_{m=0}^{\infty} \alpha_j \dots \alpha_n (\alpha_0 \dots \alpha_n)^m p_m(\alpha_0 \bar{\alpha}_0) \mathbb{k}[\alpha_0 \bar{\alpha}_0] \oplus \bigoplus_{m=0}^{\infty} \bar{\alpha}_{j-1} \dots \bar{\alpha}_0 (\bar{\alpha}_n \dots \bar{\alpha}_0)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0] \end{aligned}$$

where the first equality is Lemma 6.4 and the second equality follows by multiplying the expression for  $V_k$  on the left by  $\bar{\alpha}_{j-1}\dots\bar{\alpha}_k$  and using a similar argument as in Lemma 6.4. Here, each  $p_m$  is a polynomial in  $\alpha_0 \bar{\alpha}_0$  of degree  $j - k$ . Therefore,

$$\dim_{\mathbb{k}} (V_j/\bar{\alpha}_{j-1}\dots\bar{\alpha}_kV_k)_m = \begin{cases} j - k & \text{if } m \in (n+1)\mathbb{Z}_{\geq 1} - j \\ 0 & \text{if } m \in (n+1)\mathbb{Z}_{\leq 0} - j \end{cases}.$$

Similarly,

$$\begin{aligned} V_\ell &= \bigoplus_{m=0}^{\infty} \alpha_\ell \dots \alpha_n (\alpha_0 \dots \alpha_n)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0] \oplus \bigoplus_{m=0}^{\infty} \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_0 (\bar{\alpha}_n \dots \bar{\alpha}_0)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0] \\ \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_i V_i &= \bigoplus_{m=0}^{\infty} \alpha_\ell \dots \alpha_n (\alpha_0 \dots \alpha_n)^m q_m(\alpha_0 \bar{\alpha}_0) \mathbb{k}[\alpha_0 \bar{\alpha}_0] \oplus \bigoplus_{m=0}^{\infty} \bar{\alpha}_{\ell-1} \dots \bar{\alpha}_0 (\bar{\alpha}_n \dots \bar{\alpha}_0)^m \mathbb{k}[\alpha_0 \bar{\alpha}_0] \end{aligned}$$

where each  $q_m$  is a polynomial in  $\alpha_0 \bar{\alpha}_0$  of degree  $\ell - i = j - k$ , and so

$$\dim_{\mathbb{k}} (V_\ell/\bar{\alpha}_{\ell-1}\dots\bar{\alpha}_iV_i)_m = \begin{cases} j - k & \text{if } m \in (n+1)\mathbb{Z}_{\geq 1} - \ell \\ 0 & \text{if } m \in (n+1)\mathbb{Z}_{\leq 0} - \ell \end{cases};$$

that is

$$\dim_{\mathbb{k}} \left( (V_\ell/\bar{\alpha}_{\ell-1}\dots\bar{\alpha}_iV_i)[j - \ell] \right)_m = \begin{cases} j - k & \text{if } m \in (n+1)\mathbb{Z}_{\geq 1} - j \\ 0 & \text{if } m \in (n+1)\mathbb{Z}_{\leq 0} - j \end{cases},$$

and so the result follows.  $\square$

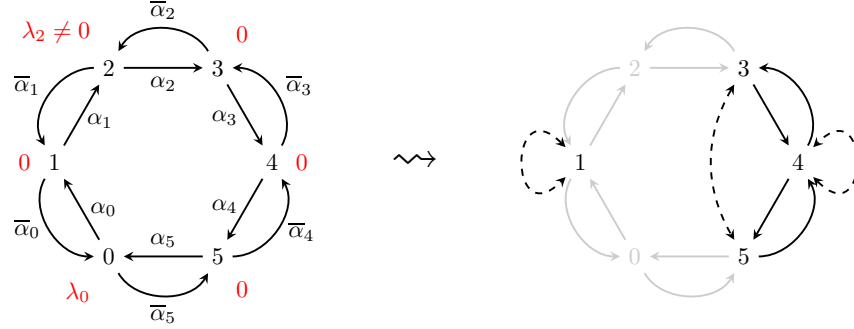
The above proposition allows us to prove that the translation functor  $\Sigma$  acts on the indecomposable objects in  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ , namely the  $V_i$  with  $i \in I_\lambda$ , as the Nakayama automorphism:

*Proof of Theorem 4.3 in type  $\mathbb{A}$ .* Let  $\lambda$  be a quasi-dominant weight for  $\tilde{Q} = \mathbb{A}_n$  and suppose that  $Q'$  is a maximal connected component of  $Q_\lambda$ , so necessarily  $Q' = \mathbb{A}_m$  for some  $m$ . Suppose that  $0 \leq i < j \leq n+1$  are the vertices of  $\tilde{Q}$  adjacent to  $Q'$ , where if we have  $j = n+1$  then this corresponds to the extending vertex 0. It is clear that the hypotheses of Proposition 6.8 hold, and so there are short exact sequences

$$0 \rightarrow V_k \rightarrow V_i \oplus V_j \rightarrow V_{i+j-k} \rightarrow 0$$

for each  $k$  with  $i < k < j$ . Since the weights at vertices  $i$  and  $j$  are nonzero, Lemma 3.3 tells us that  $V_i$  and  $V_j$  are projective  $\mathcal{O}^\lambda$ -modules, and so we have  $\Sigma V_k = V_{i+j-k}$  by definition. This means that  $\Sigma$  permutes the vertex modules corresponding to vertices of  $Q'$  as the Nakayama automorphism of  $Q'$ , as claimed.  $\square$

*Example 6.9.* Suppose that  $\tilde{Q} = \mathbb{A}_5$ , and consider the algebra  $\mathcal{O}^\lambda$  where the weight  $\lambda$  is indicated in red on the left hand quiver below, and where  $\lambda_2 \neq 0$ :



By Theorem 4.4 we have a triangle equivalence  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(Q^{(1)}) \oplus \text{proj-}\Pi(Q^{(2)})$ , where  $Q^{(1)}$  is the subquiver with vertex set  $\{1\}$ , while  $Q^{(2)}$  is the subquiver with vertex set  $\{3, 4, 5\}$ . The dashed arrows indicate how the translation functor acts on the non-projective vertex modules in  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ . If we set  $\lambda_0 = -\lambda_2$  (respectively,  $\lambda_0 = 1 - \lambda_2$ ) and write  $\lambda^c$  (respectively,  $\lambda^{nc}$ ) for the resulting weight, then  $\mathcal{O}^{\lambda^c}$  is commutative while  $\mathcal{O}^{\lambda^{nc}}$  is noncommutative, and Corollary 4.7 tells us that we have a triangle equivalence  $\underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda^c} \simeq \underline{\text{MCM}}\text{-}\mathcal{O}^{\lambda^{nc}}$ .

## 7. THE TRANSLATION FUNCTOR INDUCES A GRAPH AUTOMORPHISM

In this section we will prove that the translation functor  $\Sigma$  induces a graph automorphism of  $Q_\lambda$ , something which we have already seen to be true in the type  $\mathbb{A}$  case. Recall that there is a one-to-one correspondence between the vertices of  $Q_\lambda$  and isoclasses of indecomposable objects in  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ , with representatives given by the vertex modules whose corresponding vertex lies in  $Q_\lambda$ . Letting  $\pi$  be the permutation of the vertices induced by  $\Sigma$ , we claim that  $\pi$  is a graph automorphism. This is a very useful result which substantially reduces the amount of work that we have to do later on.

To prove this, we need the following lemma which follows quickly from results in the literature.

**Lemma 7.1.** *Let  $Q$  be a Dynkin quiver and write  $U_i = e_i \Pi(Q)$ .*

(1) *If  $Q = \mathbb{A}_n$  then*

$$\dim_{\mathbb{k}} \Pi(\mathbb{A}_n) = \frac{1}{6}n(n+1)(n+2), \quad \dim_{\mathbb{k}} U_i = i(n+1-i), \text{ for } 1 \leq i \leq n.$$

(2) *If  $Q = \mathbb{D}_n$ , where  $n \geq 4$ , then*

$$\dim_{\mathbb{k}} \Pi(\mathbb{D}_n) = \frac{1}{3}n(n-1)(2n-1),$$

$$\dim_{\mathbb{k}} U_i = 2ni - i(i+1), \text{ if } 1 \leq i \leq n-2, \quad \dim_{\mathbb{k}} U_{n-1} = \dim_{\mathbb{k}} U_n = \frac{1}{2}n(n-1).$$

(3) *If  $Q = \mathbb{E}_n$ , where  $n = 6, 7, 8$ , then the  $\mathbb{k}$ -dimensions of  $\Pi(\mathbb{E}_n) = \bigoplus_i U_i$  and the modules  $U_i$  are as follows:*

$n$	$\Pi(\mathbb{E}_n)$	$U_1$	$U_2$	$U_3$	$U_4$	$U_5$	$U_6$	$U_7$	$U_8$
6	156	22	16	30	42	30	16		
7	399	34	66	96	75	52	27	49	
8	1240	58	114	168	220	270	182	92	136

*Proof.* By [MOV06, Corollary 2.4], if  $Q$  is of type  $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_n$ , then

$$\dim_{\mathbb{k}} \Pi(Q) = \frac{1}{6}nh(h+1),$$

where  $h$  is the Coxeter number corresponding to  $Q$ , given by

$Q$	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$h$	$n+1$	$2n-2$	12	18	30



which allows one to calculate the claimed values of  $\dim_{\mathbb{k}} \Pi(Q)$ . We now calculate the dimensions of the vertex modules. Throughout let  $H(Q)$  be the matrix with  $H(Q)_{ij} = \dim_{\mathbb{k}} e_i \Pi(Q) e_j$ .

(1) By [ES98a, §4], and accounting for differences in notation, we have

$$\dim_{\mathbb{k}} e_i \Pi e_j = \begin{cases} j & \text{if } j = 1, \dots, i \\ i & \text{if } j = i + 1, \dots, n - i \\ n - j + 1 & \text{if } j = n - i + 1, \dots, n \end{cases}.$$

It follows that

$$\dim_{\mathbb{k}} U_i = \frac{1}{2}i(i+1) + i(n-2i) + \frac{1}{2}i(i+1) = i(n-i+1).$$

(2) By [ES98b, 3.4] and again adjusting for notational differences, the  $i^{\text{th}}$  row of  $H(\mathbb{D}_n)$ , where  $1 \leq i \leq n-2$ , has the form

$$\left( \begin{array}{cccccc} 2 & 4 & \dots & 2(i-1) & \underbrace{2i \quad 2i \quad \dots \quad 2i}_{n-i-1 \text{ times}} & i \quad i \end{array} \right),$$

and so

$$\dim_{\mathbb{k}} U_i = i(i-1) + 2i(n-i-1) + 2i = 2ni - i(i+1).$$

If  $i = n-1$  or  $i = n$ , then the  $i^{\text{th}}$  row instead has the form

$$\left( \begin{array}{cccccc} 1 & 2 & \dots & n-3 & n-2 & k \quad \ell \end{array} \right),$$

where  $k + \ell = n-1$ , and so  $\dim_{\mathbb{k}} U_{n-1} = \dim_{\mathbb{k}} U_n = \frac{1}{2}n(n-1)$ .

(3) Using [MOV06, Theorem 2.3.b], one can use a computer to calculate that

$$H(\mathbb{E}_6) = \begin{pmatrix} 4 & 2 & 4 & 6 & 4 & 2 \\ 2 & 2 & 3 & 4 & 3 & 2 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 6 & 4 & 8 & 12 & 8 & 4 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 2 & 2 & 3 & 4 & 3 & 2 \end{pmatrix}, \quad H(\mathbb{E}_7) = \begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 2 & 4 \\ 6 & 12 & 16 & 12 & 8 & 4 & 8 \\ 8 & 16 & 24 & 18 & 12 & 6 & 12 \\ 6 & 12 & 18 & 15 & 10 & 5 & 9 \\ 4 & 8 & 12 & 10 & 8 & 4 & 6 \\ 2 & 4 & 6 & 5 & 6 & 3 & 3 \\ 4 & 8 & 12 & 9 & 6 & 3 & 7 \end{pmatrix},$$

$$H(\mathbb{E}_8) = \begin{pmatrix} 4 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 16 & 20 & 24 & 16 & 8 & 12 \\ 8 & 16 & 24 & 30 & 36 & 24 & 12 & 18 \\ 10 & 20 & 30 & 40 & 48 & 32 & 16 & 24 \\ 12 & 24 & 36 & 48 & 60 & 40 & 20 & 30 \\ 8 & 16 & 24 & 32 & 40 & 28 & 14 & 20 \\ 4 & 8 & 12 & 16 & 20 & 14 & 8 & 10 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 16 \end{pmatrix}.$$

One can then calculate  $\dim_{\mathbb{k}} U_i$  by summing the entries in the  $i^{\text{th}}$  row of the corresponding matrix, which gives the claimed values.  $\square$

*Remark 7.2.* If we fix  $n$  in (1), then  $\dim_{\mathbb{k}} U_i$  is symmetrical about  $\frac{n+1}{2}$  when viewed as a function of  $i$ , and is strictly increasing before  $\frac{n+1}{2}$  and strictly decreasing after. Similarly, fixing  $n$  in (2), we see that  $\dim_{\mathbb{k}} U_i$  is strictly increasing when viewed as a function of  $i$ , where  $1 \leq i \leq n-2$ .

This allows us to prove the main result of this section:

**Proposition 7.3.** *The translation functor  $\Sigma$  induces a graph automorphism of  $Q_\lambda$ .*

*Proof.* First recall that  $\Sigma$  is an autoequivalence of the Krull-Schmidt category  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ , and this category has indecomposable objects given by the non-projective vertex modules; that is, vertex modules corresponding to vertices of  $Q_\lambda$ . For any such vertex module  $V_i$ ,  $\Sigma V_i = \bigoplus_{k=1}^K V_{j_k}$ , for some  $K \geq 1$  and non-projective vertex modules  $V_{j_k}$ . Applying  $\Sigma^{-1} = \Omega$  shows that  $V_i = \bigoplus_{k=1}^K \Omega(V_{j_k})$ , which, since  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$  is Krull-Schmidt, can hold only if  $K = 1$ . It follows that  $\Sigma$  permutes the non-projective vertex modules. Let  $\pi$  be the induced permutation of the vertices.

It remains to show that  $\Sigma$  is a graph automorphism. Write  $Q_\lambda = Q^{(1)} \sqcup \dots \sqcup Q^{(r)}$  as a disjoint union of connected quivers  $Q^{(j)}$ , which are therefore necessarily Dynkin. Suppose that  $i$  and  $j$  are two vertices lying in the same connected component of  $Q_\lambda$ . Then  $\dim_{\mathbb{k}} \underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Sigma V_i, \Sigma V_j) = \dim_{\mathbb{k}} \underline{\text{Hom}}_{\mathcal{O}^\lambda}(V_i, V_j) \neq 0$ , where the fact that this is nonzero follows from the equivalence  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(Q_\lambda)$ . It follows that  $\Sigma V_i$  and  $\Sigma V_j$  lie in the same connected component of  $Q_\lambda$ .

Finally, consider a vertex  $i$  in a connected component  $Q^{(k)}$ . Assume that  $\pi(i)$  lies in  $Q^{(\ell)}$ . Then, using the equivalence  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(Q_\lambda)$ ,

$$\begin{aligned} \dim_{\mathbb{k}} e_i \Pi(Q^{(k)}) &= \dim_{\mathbb{k}} e_i \Pi(Q_\lambda) \\ &= \dim_{\mathbb{k}} \underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, V_i) \\ &= \dim_{\mathbb{k}} \underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Sigma(\Pi^\lambda e_0), \Sigma V_i) \\ &= \dim_{\mathbb{k}} \underline{\text{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, V_{\pi(i)}) \\ &= \dim_{\mathbb{k}} e_{\pi(i)} \Pi(Q_\lambda) \\ &= \dim_{\mathbb{k}} e_{\pi(i)} \Pi(Q^{(\ell)}). \end{aligned}$$

Since  $\pi$  sends connected components to connected components, combining this fact with Lemma 7.1 and the remark following it, we see that this is only possible if the permutation  $\pi$  induced by  $\Sigma$  is a graph automorphism of  $Q_\lambda$ .  $\square$

*Remark 7.4.* If  $Q_\lambda$  has two connected components which are the same then, a priori, the induced graph automorphism could interchange these components. It turns out that  $\Sigma$  actually maps a connected component of  $Q_\lambda$  back to itself. We have already seen this to be true in the type  $\mathbb{A}$  case, and for types  $\mathbb{D}$  and  $\mathbb{E}$  this will follow from the results of Sections 9 and 10.

## 8. THE KNITTING ALGORITHM

We now wish to determine the syzygy functor in the type  $\mathbb{D}$  and type  $\mathbb{E}$  cases. In the type  $\mathbb{A}$  case we were able to exploit the fact that  $\mathcal{O}^\lambda$  had the structure of a  $\mathbb{Z}$ -graded ring for any  $\lambda$ , but this is not the case in types  $\mathbb{D}$  and type  $\mathbb{E}$ . Instead, we identify a number of short exact sequences in the case when  $\lambda = \mathbf{0}$  and show that these remain exact for other weights provided certain hypotheses are met.

In [IW10], Iyama and Wemyss outline an algorithm to compute short exact sequences of maximal Cohen-Macaulay  $\mathbb{k}[[x, y]]^G$ -modules, where  $G$  is a finite subgroup of  $\text{SL}(2, \mathbb{C})$ . We consider only the cases where  $G$  isn't cyclic (that is, we consider only the type  $\mathbb{D}$  and  $\mathbb{E}$  cases). Fix such a  $G$ , let  $\tilde{Q}$  be the quiver obtained from its associated extended Dynkin graph by assigning an orientation to the edges so that its arrows coincide with the ordinary arrows in Figure 1, and write  $R = \mathcal{O}(\tilde{Q})$ . If we filter  $R$  (and hence also the vertex modules  $V_i$ ) by path length, then, letting  $(\hat{\cdot})$  denote the completion of a ring or module with respect to this filtration, we have  $\hat{R} \cong \mathbb{k}[[x, y]]^G$  and  $\text{MCM-}\hat{R} = \text{add}(\bigoplus_i \hat{V}_i)$ . Now construct the repetition quiver of  $\tilde{Q}$  (c.f. [Hap88, 5.6]) to the left but omitting the labelling of the vertices and arrows, as in Figure 2. We refer to this quiver

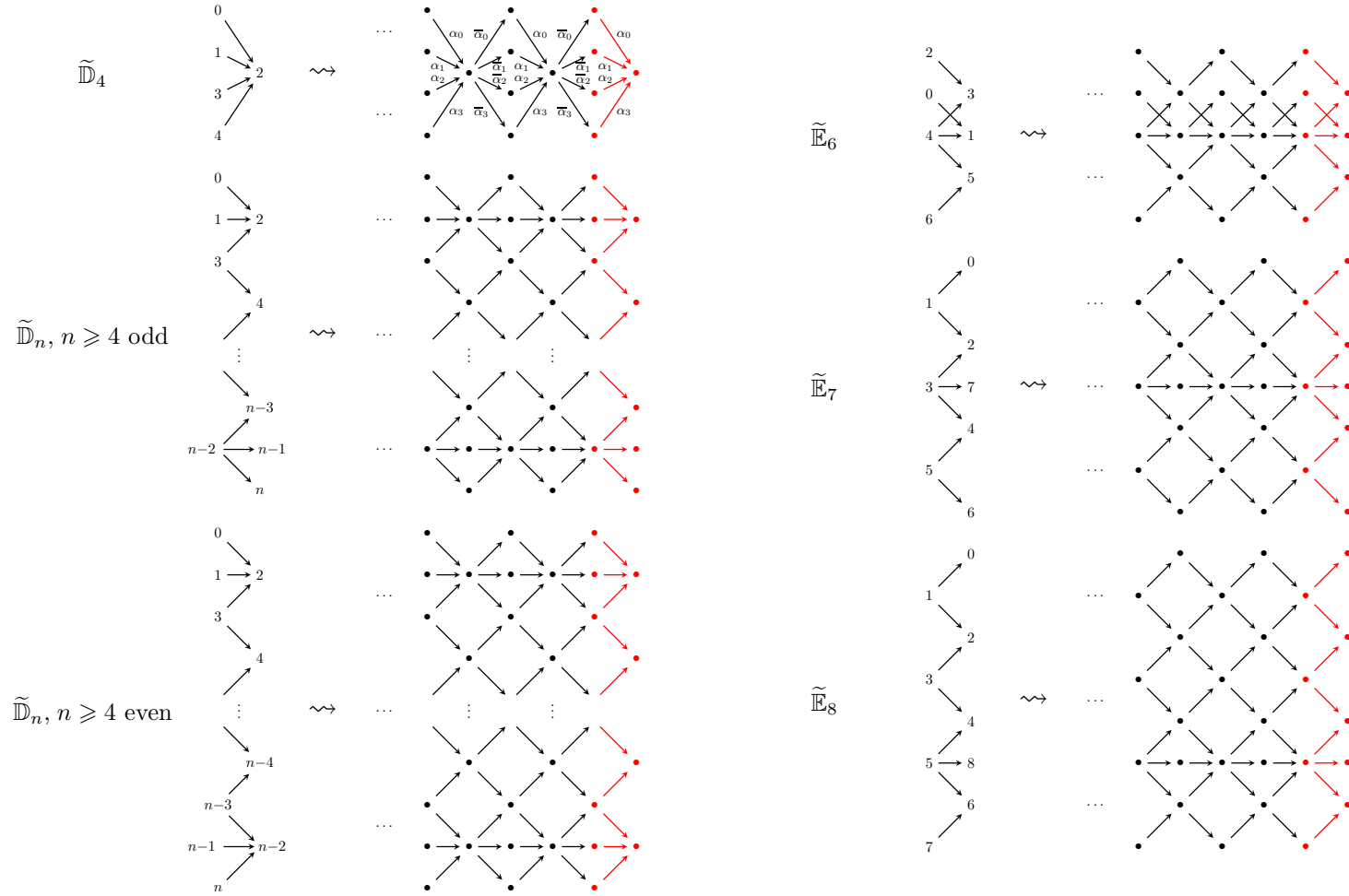


FIGURE 2. The extended Dynkin quivers of types  $\mathbb{D}$  and  $\mathbb{E}$  and their associated left-infinite repetition quivers. The red arrows and vertices of each repetition quiver correspond to the first copy of  $\tilde{Q}$ . To aid the reader, we label each arrow of  $\text{rep}(\tilde{\mathbb{D}}_4)$  to illustrate which arrow in the double of  $\mathbb{D}_4$  it corresponds to; the other cases are similar.

as  $\text{rep}(\tilde{Q})$  since it is built out of repeated copies of  $\tilde{Q}$ . We will sometimes have need to speak of the  $k^{\text{th}}$  copy of  $\tilde{Q}$  appearing in  $\text{rep}(\tilde{Q})$ , reading from right to left.

In each copy of  $\tilde{Q}$  in  $\text{rep}(\tilde{Q})$ , associate to each vertex the corresponding vertex module, to each arrow the corresponding ordinary arrow in the double of  $\tilde{Q}$ , and to each arrow between the copies of  $\tilde{Q}$  the corresponding reverse arrow (an example of this correspondence is illustrated in Figure 2). We are now in a position to describe the *knitting algorithm*.

Let  $S$  be a subset of the vertex modules which contains  $\hat{V}_0 = \hat{R}$ . The knitting algorithm explicitly constructs a surjection (in fact, there are a number of valid choices for this surjection) from a direct sum of copies of those modules in  $S$  to another chosen vertex module which is not in  $S$ . Moreover, the algorithm also identifies the kernel of this surjection; it is necessarily isomorphic to one of the vertex modules not in  $S$ . The algorithm is as follows:

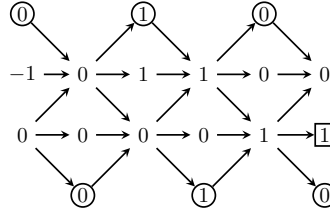
- (1) In  $\text{rep}(\tilde{Q})$ , circle each vertex which corresponds to a vertex module in  $S$ .
- (2) Suppose we wish to surject onto  $\hat{V}_i$ . Place a 1 at vertex  $i$  in the first copy of  $\tilde{Q}$ , box it, and place zeros above and below it in its column.
- (3) If  $i$  appears in the first column from the right, then for each vertex  $j$  in the second column place a 1 at that vertex if there is an arrow  $j \rightarrow i$  in  $\text{rep}(\tilde{Q})$ , and place zeros at the remaining positions in the second column. If instead  $i$  appears in the second column from the right, place zeros above and below it in its column, and also at each vertex in the first column.
- (4) Assuming the first  $\ell$  columns are filled, where  $\ell \geq 2$ , fill the  $(\ell + 1)^{\text{st}}$  column as follows: the value at a vertex  $k$  in column  $\ell + 1$  is equal to the sum of all the values at uncircled vertices  $j$  in column  $\ell$  with an arrow  $k \rightarrow j$ , minus the number at the vertex corresponding to vertex  $k$  in column  $\ell - 1$ , provided that it is uncircled.
- (5) Repeat step (4) until a  $-1$  is placed at a vertex, and then stop.

For each vertex  $j \in S$ , let  $a_j$  be the sum of the values in the (necessarily circled) vertices corresponding to vertex  $j$ . Let  $M = \bigoplus_{j \in S} \hat{V}_j^{\oplus a_j}$ . Suppose that  $i'$  is the vertex in  $\tilde{Q}$  which corresponds to where the  $-1$  was placed. We then have a short exact sequence

$$0 \rightarrow \hat{V}_{i'} \xrightarrow{\phi} M \xrightarrow{\psi} \hat{V}_i \rightarrow 0. \quad (8.1)$$

We will shortly explain how to determine  $\phi$  and  $\psi$ . We often refer to a completed diagram arising from the knitting algorithm as a (*knitting*) *pattern*. The following example illustrates the algorithm:

*Example 8.2.* Suppose that  $R = \mathcal{O}(\mathbb{D}_5)$ , and consider its completion  $\hat{R} = \mathbb{k}[[x, y]]^G$ , where  $G$  is the binary dihedral group of order  $4 \cdot (5 - 2) = 12$ . Suppose that  $S = \{\hat{V}_0, \hat{V}_5\}$  and that we wish to surject onto  $\hat{V}_4$ . Following the steps outlined above in the knitting algorithm, the completed knitting pattern is:



From this, one sees that we have a short exact sequence

$$0 \rightarrow \hat{V}_1 \xrightarrow{\phi} \hat{V}_0 \oplus \hat{V}_5 \xrightarrow{\psi} \hat{V}_4 \rightarrow 0.$$

The following lemma shows that these sequences remain exact in the non-complete setting:

**Lemma 8.3.** *Let  $R = \mathcal{O}(\tilde{Q})$ , filter it by path length, and consider its completion  $\hat{R} \cong \mathbb{k}[[x, y]]^G$  with respect to this filtration. If*

$$0 \rightarrow \hat{V}_{i'} \rightarrow \bigoplus_{j \in S} \hat{V}_j^{\oplus a_j} \rightarrow \hat{V}_i \rightarrow 0$$

*is an exact sequence of MCM  $\hat{R}$ -modules obtained via the knitting algorithm, then*

$$0 \rightarrow V_{i'} \rightarrow \bigoplus_{j \in S} V_j^{\oplus a_j} \rightarrow V_i \rightarrow 0$$

*is an exact sequence of  $R$ -modules.*

*Proof.*  $\hat{R}$  has a filtration  $(F_i)_{i \geq 0}$  where  $F_i$  consists of paths of length at most  $i$ , and with this filtration we have  $\text{gr } \hat{R} = \mathcal{O}$ . Moreover, the induced filtration on the indecomposable maximal Cohen-Macaulay modules  $\hat{V}_i$  satisfies  $\text{gr } \hat{V}_i = V_i$ . Applying the functor  $\text{gr}$ , which is exact by [MR01, Proposition 7.6.13], to these short exact sequences of  $\hat{R}$ -modules completes the proof.  $\square$

In later sections, we will need to explicitly determine the maps appearing in the short exact sequences arising from the knitting algorithm; for example, we would like to know how to determine the maps  $\phi$  and  $\psi$  from Example 8.2. From now on we shall assume that we are working in  $R = \mathcal{O}(\tilde{Q})$  rather than  $\hat{R}$ , which is justified by Lemma 8.3, and so maps between vertex modules correspond to elements of  $\Pi(\tilde{Q})$ .

So suppose that we obtain a short exact sequence

$$0 \rightarrow V_{i'} \xrightarrow{\phi} \bigoplus_{j \in S} V_j^{\oplus a_j} \xrightarrow{\psi} V_i \rightarrow 0 \quad (8.4)$$

from the knitting algorithm. Determining a valid choice for  $\psi$  is relatively straightforward. Suppose we wish to determine the  $V_j \rightarrow V_i$  component of  $\psi$ , where  $V_j$  corresponds to a circled vertex in the knitting pattern corresponding to vertex  $j$  in  $\tilde{Q}$ . Between this vertex and the boxed vertex corresponding to vertex  $i$  in the knitting pattern, there will necessarily be a path which visits only vertices which have a nonzero label and which are not circled (often there are multiple such paths; any choice is valid). This corresponds to a path  $p$  from vertex  $j$  to vertex  $i$  in  $\tilde{Q}$ . Taking the reverse of this path gives us a valid choice for the  $V_j \rightarrow V_i$  component of  $\psi$  (for example, if  $p = \alpha_2 \bar{\alpha}_1 \alpha_0$ , then its reverse is  $\bar{\alpha}_0 \alpha_1 \bar{\alpha}_2$ ).

Determining  $\phi$  is slightly more involved. First observe that we can grade everything by path length. Now suppose we wish to determine the  $V_{i'} \rightarrow V_j$  component of  $\phi$ , where  $i'$  corresponds to the vertex labelled by  $-1$  and  $j$  corresponds to a circled vertex containing a  $1$ . Any path in the knitting pattern between these vertices has the same length, and this determines the degree of the corresponding component of  $\phi$ . However, one cannot arbitrarily choose a path between these vertices and expect to get a valid choice for this component, as was essentially the case for  $\psi$  (although in many situations there is only a single such path); at the very least, we need to ensure that the composition  $\psi\phi$  is zero. The following argument, due to Wemyss, explains that this condition is in fact sufficient:

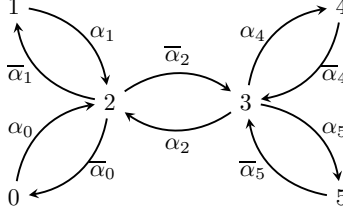
**Lemma 8.5.** *Suppose that (8.4) is a short exact sequence arising from the knitting algorithm, and that  $\psi$  has been determined as above. Suppose that  $\tilde{\phi} : V_{i'} \rightarrow \bigoplus_{j \in S} V_j^{\oplus a_j}$  is such that the degree of each component is equal to the degree of the corresponding component of  $\phi$ , as prescribed by the knitting algorithm, and that  $\psi\tilde{\phi} = 0$ . Then  $\phi = \tilde{\phi}$ .*

*Proof.* The knitting algorithm tells us that  $\ker \psi \cong V_{i'}$ , so by the universal property of kernels in the category of graded modules, there exists a graded morphism  $\theta : V_{i'} \rightarrow \ker \psi \cong V_{i'}$  with

$\phi\theta = \tilde{\phi}$ . Since  $\phi$  and  $\tilde{\phi}$  have the same degree, this forces  $\theta$  to have degree 0. But the only degree 0 endomorphism of  $V_{i'}$  is the identity, so  $\phi = \tilde{\phi}$ .  $\square$

We now demonstrate the usefulness of the above lemma.

*Example 8.6.* We return to Example 8.2 where we wish to determine the maps  $\phi$  and  $\psi$ . For the reader's convenience, we remind them of our labelling of the double of  $\tilde{Q}$ .



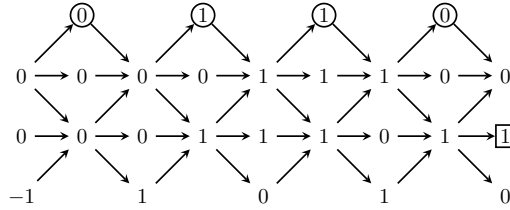
There are unique paths in the completed knitting pattern from the circled 1's at vertices 0 and 5 in the second copy of  $\tilde{Q}$  to vertex 4 (which is boxed) in the first copy of  $\tilde{Q}$ . These paths are identified with paths in the double of  $\tilde{Q}$ , and taking their reverses yields the elements  $\bar{\alpha}_4\alpha_2\bar{\alpha}_0$  and  $\bar{\alpha}_4\alpha_5$ , which give rise to maps  $V_0 \rightarrow V_4$  and  $V_5 \rightarrow V_4$ , respectively. To determine  $\phi$ , observe that there are unique shortest paths from the  $-1$  at vertex 1 to the aforementioned circled 1's, and these correspond to the elements  $\alpha_0\bar{\alpha}_1$  and  $\bar{\alpha}_5\alpha_2\bar{\alpha}_1$ . Direct calculation using the relations in  $\mathcal{O}(\tilde{\mathbb{D}}_5)$  shows that

$$\bar{\alpha}_4\alpha_2\bar{\alpha}_0 \cdot \alpha_0\bar{\alpha}_1 - \bar{\alpha}_4\alpha_5 \cdot \bar{\alpha}_5\alpha_2\bar{\alpha}_1 = -\bar{\alpha}_4\alpha_2\bar{\alpha}_2 \cdot \alpha_2\bar{\alpha}_1 + \bar{\alpha}_4\alpha_2 \cdot \bar{\alpha}_2\alpha_2\bar{\alpha}_1 = 0,$$

and so, viewing direct sums as column vectors,  $\phi$  and  $\psi$  are, for example, given by left multiplication by the matrices

$$\phi = \begin{pmatrix} \alpha_0\bar{\alpha}_1 \\ \bar{\alpha}_5\alpha_2\bar{\alpha}_1 \end{pmatrix}, \quad \psi = (\bar{\alpha}_4\alpha_2\bar{\alpha}_0 \quad -\bar{\alpha}_4\alpha_5).$$

For an example where we need to make use of Lemma 8.5, keep  $\tilde{Q} = \tilde{\mathbb{D}}_5$ , but now suppose that  $S = \{V_0\}$ . To find a short exact sequence ending with  $V_4$ , we use the knitting algorithm to obtain the following pattern:



We are able to read off the short exact sequence

$$0 \rightarrow V_5 \xrightarrow{\phi} V_0^{\oplus 2} \xrightarrow{\psi} V_4 \rightarrow 0.$$

We now determine the components of  $\psi$ . The path in the knitting pattern from the circled 1 in the fourth column from the right to the boxed 1 corresponds to the morphism  $V_0 \rightarrow V_4$  given by left multiplication by  $\bar{\alpha}_4\alpha_2\bar{\alpha}_0$ . On the other hand, there are multiple paths which pass only through vertices labelled by uncircled 1's from the circled 1 in the sixth column to the boxed 1. One such path goes through vertices 0, 2, 3, 5, 3, 4, in that order, and this corresponds to the morphism  $V_0 \rightarrow V_4$  given by left multiplication by  $\bar{\alpha}_4\alpha_5\bar{\alpha}_5\alpha_2\bar{\alpha}_0$ .

We can also immediately determine one of the components of  $\phi$ , up to a change of sign. There

is a unique path in the knitting pattern from the vertex labelled with  $-1$  to the circled  $1$  in the sixth column from the right, so the component  $V_5 \rightarrow V_0$  is given by left multiplication by  $\alpha_0 \bar{\alpha}_2 \alpha_5$ . We now calculate that (where here to improve readability we indicate in red the terms which are changed using the relations in  $\Pi(\tilde{Q})$  at each step in the sequences of equalities)

$$\begin{aligned}
& \bar{\alpha}_4 \alpha_5 \bar{\alpha}_5 \alpha_2 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_2 \alpha_5 + \bar{\alpha}_4 \alpha_2 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_5 \\
&= -\bar{\alpha}_4 \alpha_5 \bar{\alpha}_5 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_5 - \bar{\alpha}_4 \alpha_5 \bar{\alpha}_5 \alpha_2 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_5 - \bar{\alpha}_4 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_5 - \bar{\alpha}_4 \alpha_2 \bar{\alpha}_1 \underbrace{\alpha_1 \bar{\alpha}_1}_{=0} \alpha_1 \bar{\alpha}_2 \alpha_5 \\
&= \bar{\alpha}_4 \alpha_5 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_4 \alpha_2 \bar{\alpha}_2 \alpha_5 + \bar{\alpha}_4 \alpha_5 \underbrace{\bar{\alpha}_5 \alpha_5}_{=0} \bar{\alpha}_5 \alpha_2 \bar{\alpha}_2 \alpha_5 + \bar{\alpha}_4 (\alpha_5 \bar{\alpha}_5 + \alpha_2 \bar{\alpha}_2) \alpha_2 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_5 \\
&= -\bar{\alpha}_4 \alpha_5 \bar{\alpha}_5 \alpha_4 \underbrace{\bar{\alpha}_4 \alpha_4}_{=0} \bar{\alpha}_4 \alpha_5 - \bar{\alpha}_4 \alpha_5 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_4 \underbrace{\alpha_5 \bar{\alpha}_5}_{=0} \alpha_5 - \bar{\alpha}_4 \underbrace{\alpha_4 \bar{\alpha}_4}_{=0} \alpha_2 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_5 \\
&= 0,
\end{aligned}$$

and so Lemma 8.5 tells us that possible choices for the maps in the short exact sequence are

$$\phi = \begin{pmatrix} \alpha_0 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_5 \\ \alpha_0 \bar{\alpha}_2 \alpha_5 \end{pmatrix}, \quad \psi = (\bar{\alpha}_4 \alpha_2 \bar{\alpha}_0 \quad \bar{\alpha}_4 \alpha_5 \bar{\alpha}_5 \alpha_2 \bar{\alpha}_0).$$

When doing calculations similar to those above in later sections, we will omit some of the steps for the sake of brevity. We will also use colour in the same way to improve readability.

In the next two sections, we use the knitting algorithm to determine the syzygy functor in the type  $\mathbb{D}$  and  $\mathbb{E}$  cases. Since Lemma 8.3 shows that the knitting algorithm is valid for  $R = \mathcal{O}(\tilde{Q})$  and not just its completion, we need not assume that we are in the complete local setting.

We now briefly outline how we will go about proving Theorem 4.3 in these cases. In Section 3 we saw that, given a quasi-dominant weight  $\lambda$ , the singularity category of  $\mathcal{O}^\lambda$  is Krull-Schmidt with an indecomposable object for each vertex of  $Q_\lambda$ , namely the (isoclass of) those vertex modules which are not projective  $\mathcal{O}^\lambda$ -modules, which correspond to those vertices  $i \in I_\lambda$ .

We will first identify all possible connected subquivers of  $Q$ , which will necessarily be Dynkin. Given such a subquiver  $Q'$ , let  $\partial Q'$  be the set of all vertices of  $\tilde{Q}$  which are adjacent to  $Q'$ . Given a vertex module  $V_i$  with  $i \in Q'_0$ , we can use the knitting algorithm in  $R := \mathcal{O}(\tilde{Q})$  to find a short exact sequence of  $R$ -modules whose right-hand term is  $V_i$  and whose middle term is a direct sum of vertex modules whose corresponding vertices lie in  $\partial Q'$ . We then show that such a sequence remains exact in  $\text{mod-}\mathcal{O}^\lambda$  provided that  $Q'$  is one of the connected components of  $Q_\lambda$ . This will allow us to determine how  $\Sigma$  behaves on a subset of the indecomposable objects of  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ . This subset is chosen so that, using Proposition 7.3, we are able to determine  $\Sigma$  on all the indecomposable objects of  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ <sup>2</sup>. This will complete the proof of Theorem 4.3.

We end this section with a basic lemma which will be used in the following two sections.

**Lemma 8.7.** *Let  $S$  be a filtered ring and let*

$$M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3 \tag{8.8}$$

*be a complex of filtered modules and filtered homomorphisms. Suppose that*

$$\text{gr } M_1 \xrightarrow{\text{gr } \phi} \text{gr } M_2 \xrightarrow{\text{gr } \psi} \text{gr } M_3 \tag{8.9}$$

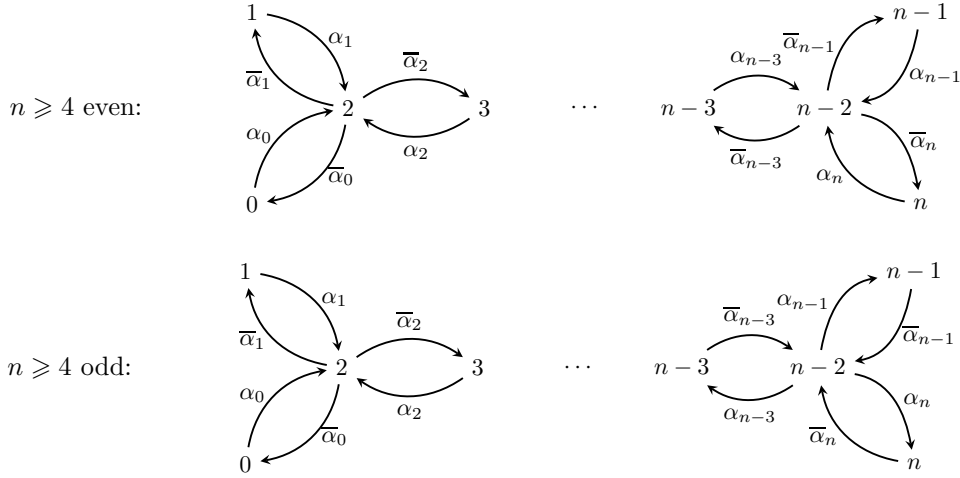
*is an exact sequence of  $\text{gr } S$ -modules. Then (8.8) is exact.*

<sup>2</sup>For every subquiver  $Q'$  that isn't of type  $\mathbb{D}_4$ ,  $\mathbb{E}_7$ , or  $\mathbb{E}_8$ , we only need to determine  $\Sigma V_i$  for one carefully chosen vertex  $i \in Q'$ . If  $Q' = \mathbb{D}_4$ , we need to determine  $\Sigma$  on two vertex modules, while if  $Q'$  is  $\mathbb{E}_7$ , or  $\mathbb{E}_8$  then, since their automorphism groups are trivial, and a Dynkin quiver can have at most one subquiver of type  $\mathbb{E}_7$  or  $\mathbb{E}_8$  (c.f. Remark 7.4),  $\Sigma$  is immediately determined by Proposition 7.3.

*Proof.* Since (8.8) is a complex, we have  $\text{im } \phi \subseteq \ker \psi$  and so  $\text{gr im } \phi \subseteq \text{gr ker } \psi$ . It is always true that  $\text{gr ker } \psi \subseteq \ker \text{gr } \psi$  and  $\text{im gr } \phi \subseteq \text{gr im } \phi$ , and combining this with that fact that  $\ker \text{gr } \psi = \text{im gr } \phi$  (by exactness of (8.9)), it follows that  $\text{gr ker } \psi \subseteq \text{gr im } \phi$ . Therefore  $\text{gr im } \phi = \text{gr ker } \psi$ . But also  $\text{im } \phi \subseteq \ker \psi$ , and it is standard (c.f. [MR01, Proposition 1.6.7]) that together these facts imply  $\text{im } \phi = \ker \psi$ .  $\square$

## 9. THE TRANSLATION FUNCTOR IN TYPE $\mathbb{D}$

Let  $\tilde{Q}$  be an extended Dynkin quiver of type  $\tilde{\mathbb{D}}_n$ , let  $Q$  be the quiver obtained by removing the extending vertex, and write  $R = \mathcal{O}(\tilde{Q})$ . We label the vertices and arrows of the double of  $Q$  as in Figure 1, which for the reader's convenience we repeat below:



Following the proof outline preceding Lemma 8.7, we now identify all possible connected subquivers which arise by deleting vertices from  $Q$ . This is easy to do:

- (Type  $\mathbb{A}$ ) We consider four different ways of obtaining a type  $\mathbb{A}$  subquiver. There is always an  $\mathbb{A}_3$  subquiver with vertices  $n-2, n-1, n$ . We can also obtain  $\mathbb{A}_{n-i}$  subquivers in two symmetrical ways, by considering the vertices  $i, i+1, \dots, n-2, n-1$ , or the vertices  $i, i+1, \dots, n-2, n$ , where  $1 \leq i \leq n-1$ . Finally, we can obtain  $\mathbb{A}_{j-i+1}$  subquivers by considering the vertices  $i, i+1, \dots, j-1, j$ , where  $1 \leq i \leq j \leq n-2$ .
- (Type  $\mathbb{D}$ ) One can obtain  $\mathbb{D}_{n-i}$  subquivers, where  $0 \leq i \leq n-4$ , by considering the vertices  $i+1, i+2, \dots, n-2, n-1, n$ .

We now wish to use the knitting algorithm to identify a number of short exact sequences of  $R$ -modules. The following proposition is split into two parts, corresponding to the different families of Dynkin subquivers. After stating the proposition, we will explain how these short exact sequences correspond to the Dynkin subquivers that we identified above.

The proof of this proposition makes use of the knitting algorithm. Following the lead of [IW10], we only prove the existence of these exact sequences for some specific parameter values; the proofs for the other members of a family of short exact sequences are essentially the same, and so for practical purposes we only outline the general pattern that one obtains from the knitting algorithm.

**Proposition 9.1.** *In the following, we work over  $R = \mathcal{O}(\tilde{Q})$ , where  $\tilde{Q} = \tilde{D}_n$  for some  $n \geq 4$ .*



(1) (Type  $\mathbb{A}$  subquivers) There are short exact sequences

- (a)  $0 \rightarrow V_3 \rightarrow V_0 \oplus V_1 \rightarrow V_4 \rightarrow 0$  if  $n = 4$   
 $0 \rightarrow V_{n-1} \rightarrow V_{n-3} \rightarrow V_n \rightarrow 0$  if  $n \neq 4$   
 (b)  $0 \rightarrow V_i \rightarrow V_{i-1} \oplus V_{m'} \rightarrow V_m \rightarrow 0$  where  $1 \leq i \leq n-2, i \neq 2$   
 $0 \rightarrow V_i \rightarrow V_0 \oplus V_1 \oplus V_{m'} \rightarrow V_m \rightarrow 0$  where  $i = 2$   
 $0 \rightarrow V_m \rightarrow V_{n-2} \rightarrow V_m \rightarrow 0$   
 where here  $(m, m') = (n-1, n)$  or  $(m, m') = (n-1, n)$ ,  
 (c)  $0 \rightarrow V_i \rightarrow V_{i-1} \oplus V_{n-1} \oplus V_n \rightarrow V_{n-2} \rightarrow 0$  where  $1 \leq i \leq n-2, i \neq 2$   
 $0 \rightarrow V_i \rightarrow V_0 \oplus V_1 \oplus V_{n-1} \oplus V_n \rightarrow V_{n-2} \rightarrow 0$  where  $i = 2$   
 $0 \rightarrow V_i \rightarrow V_{i-1} \oplus V_{j+1} \rightarrow V_j \rightarrow 0$  where  $1 \leq i \leq j \leq n-3, i \neq 2, (i, j) \neq (1, 1)$   
 $0 \rightarrow V_i \rightarrow V_0 \oplus V_1 \oplus V_{j+1} \rightarrow V_j \rightarrow 0$  where  $2 \leq j \leq n-3, i = 2$   
 $0 \rightarrow V_i \rightarrow V_2 \rightarrow V_j \rightarrow 0$  where  $(i, j) = (1, 1)$

(2) (Type  $\mathbb{D}$  subquivers) Suppose that  $0 \leq i \leq n-4$ . Then there are short exact sequences

- (a)  $0 \rightarrow V_m \rightarrow V_0^{\oplus 2} \rightarrow V_m \rightarrow 0$  if  $i = 0$   
 $0 \rightarrow V_m \rightarrow V_0 \oplus V_1 \rightarrow V_m \rightarrow 0$  if  $i = 1$   
 $0 \rightarrow V_m \rightarrow V_i \rightarrow V_m \rightarrow 0$  if  $2 \leq i \leq n-4$

if  $n-i$  is even, where  $m = n-1$  or  $m = n$ ; and

- (b)  $0 \rightarrow V_{m'} \rightarrow V_0^{\oplus 2} \rightarrow V_m \rightarrow 0$  if  $i = 0$   
 $0 \rightarrow V_{m'} \rightarrow V_0 \oplus V_1 \rightarrow V_m \rightarrow 0$  if  $i = 1$   
 $0 \rightarrow V_{m'} \rightarrow V_i \rightarrow V_m \rightarrow 0$  if  $2 \leq i \leq n-4$

if  $n-i$  is odd, where  $(m, m') = (n-1, n)$  or  $(m, m') = (n, n-1)$ .

In (1), a map between vertex modules  $V_k$  and  $V_\ell$  is given by left multiplication by the shortest path from vertex  $\ell$  to vertex  $k$ , possibly with a change of sign. The same is true in (2), except for the cases where  $i = 0$ , and in this case the maps are given in (9.12).

*Remark 9.2.* We now describe how these short exact sequences correspond to the Dynkin subquivers we identified. Given a short exact sequence  $\mathcal{S}$  from Proposition 9.1, write  $W_f$  (respectively,  $W_m$ ) for the set of vertices corresponding to the vertex modules appearing in the flanking terms (respectively, the middle term) of  $\mathcal{S}$ . Then  $\mathcal{S}$  corresponds to the unique subquiver  $Q'$  of  $Q$  with  $W_f \subseteq Q'_0$  and with  $W_m \subseteq \partial Q'$ , where  $\partial Q'$  is the set of all vertices of  $\tilde{Q}$  which are adjacent to  $Q'$ . In part (1), there is a one-to-one correspondence between short exact sequences and type  $\mathbb{A}$  subquivers, while in part (2) there is a similar correspondence which is instead two-to-one. The fact that we have a two-to-one correspondence turns out to be important only when the subquiver is of type  $\mathbb{D}_4$ , stemming from the fact that  $|\text{Aut}(\mathbb{D}_4)| > 2$ .

*Proof of Proposition 9.1.* Throughout the proof, given a short exact sequence of  $R$ -modules, we shall write  $\phi$  and  $\psi$  for the corresponding injection and surjection, respectively. Bullets appearing in a knitting pattern will indicate unimportant zeros, and we omit the arrows.

(1) The exact sequences in (a) follow from the three patterns below, where the left hand pattern corresponds to the first exact sequence, and the middle and right hand patterns correspond to the second exact sequence in the cases when  $n$  is even and odd, respectively:

$$\begin{array}{ccc}
 \begin{array}{ccc} \textcircled{0} & \textcircled{1} & \textcircled{0} \\ \textcircled{0} & \textcircled{1} & \textcircled{0} \\ & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array} & 
 \begin{array}{ccc} \vdots & \vdots & \vdots \\ & 0 & 0 & 0 \\ \textcircled{0} & \textcircled{1} & \textcircled{0} \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} \end{array} & 
 \begin{array}{ccc} \vdots & \vdots & \\ & 0 & 0 \\ \textcircled{0} & \textcircled{1} & \textcircled{0} \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}
 \end{array}$$

These patterns imply the existence of short exact sequences

$$0 \rightarrow V_3 \rightarrow V_0 \oplus V_1 \rightarrow V_4 \rightarrow 0 \quad (9.3)$$

$$0 \rightarrow V_{n-1} \rightarrow V_{n-3} \rightarrow V_n \rightarrow 0 \quad (9.4)$$

where  $n = 4$  in (9.3) and  $n \geq 5$  in (9.4). In each case, using Lemma 8.5 we find that the morphisms between vertex modules correspond to the shortest path between the corresponding vertices. For example, for the first exact sequence if we set

$$\phi = \begin{pmatrix} \alpha_0 \bar{\alpha}_3 \\ \alpha_1 \bar{\alpha}_3 \end{pmatrix} \quad \psi = (\alpha_4 \bar{\alpha}_0 \quad \alpha_4 \bar{\alpha}_1)$$

then one calculates that

$$\begin{aligned} \phi\psi &= \alpha_4 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_3 + \alpha_4 \bar{\alpha}_1 \cdot \alpha_1 \bar{\alpha}_3 = \alpha_4 (\bar{\alpha}_0 \alpha_0 + \bar{\alpha}_1 \alpha_1) \bar{\alpha}_3 = -\alpha_4 (\bar{\alpha}_3 \alpha_3 + \bar{\alpha}_4 \alpha_4) \bar{\alpha}_3 \\ &= -\alpha_4 \bar{\alpha}_3 \underbrace{\alpha_3 \bar{\alpha}_3}_{=0} - \underbrace{\alpha_4 \bar{\alpha}_4}_{=0} \alpha_4 \bar{\alpha}_3 = 0 \end{aligned} \quad (9.5)$$

and so these are valid choices for the maps. Similarly, for the second short exact sequence (assuming  $n$  is even) if we set

$$\phi = \alpha_{n-3} \bar{\alpha}_{n-1} \quad \psi = \alpha_n \bar{\alpha}_{n-3}$$

then we find that

$$\begin{aligned} \psi\phi &= \alpha_n \bar{\alpha}_{n-3} \cdot \alpha_{n-3} \bar{\alpha}_{n-1} = -\alpha_n (\bar{\alpha}_{n-1} \alpha_{n-1} + \bar{\alpha}_n \alpha_n) \bar{\alpha}_{n-1} \\ &= -\alpha_n \bar{\alpha}_{n-1} \underbrace{\alpha_{n-1} \bar{\alpha}_{n-1}}_{=0} - \underbrace{\alpha_n \bar{\alpha}_n}_{=0} \alpha_n \bar{\alpha}_{n-1} = 0, \end{aligned} \quad (9.6)$$

and again these are suitable choices for the maps.

We now consider the families of short exact sequences in (b). We shall assume that  $m = n$ , with the case  $m = n - 1$  being symmetrical. We provide the knitting patterns for  $n = 10$  and  $i = 1, 2, 3$  below, before describing the general pattern. They are, respectively:

$$\begin{array}{ccc} \begin{array}{cccccccc} \textcircled{0} & \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccccccc} \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \textcircled{1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{array} & \begin{array}{cccccccc} \textcircled{0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{array} \end{array}$$

From these we read off the claimed short exact sequences

$$\begin{aligned} 0 &\rightarrow V_1 \rightarrow V_0 \oplus V_9 \rightarrow V_{10} \rightarrow 0 \\ 0 &\rightarrow V_2 \rightarrow V_0 \oplus V_1 \oplus V_9 \rightarrow V_{10} \rightarrow 0 \\ 0 &\rightarrow V_3 \rightarrow V_2 \oplus V_9 \rightarrow V_{10} \rightarrow 0 \end{aligned} \quad (9.7)$$

and, using Lemma 8.5, we observe that the maps between two vertex modules are, up to sign, given by the shortest path between the corresponding vertices. We write down the maps and verify that their composition is 0 only for the case  $i = 2$  (corresponding to (9.7)), which is marginally more involved than when  $i = 1$  or  $i = 3$ . In this case, if we set

$$\phi = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_9 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \end{pmatrix} \quad \psi = (\alpha_{10} \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_0 \quad \alpha_{10} \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_1 \quad \alpha_{10} \bar{\alpha}_9)$$

then one calculates

$$\begin{aligned}
\psi\phi &= \alpha_{10}\bar{\alpha}_7\alpha_6\bar{\alpha}_5\alpha_4\bar{\alpha}_3\alpha_2(\bar{\alpha}_0\alpha_0 + \bar{\alpha}_1\alpha_1) + \alpha_{10}\bar{\alpha}_9 \cdot \alpha_9\bar{\alpha}_7\alpha_6\bar{\alpha}_5\alpha_4\bar{\alpha}_3\alpha_2 \\
&= \alpha_{10}\bar{\alpha}_7\alpha_6\bar{\alpha}_5\alpha_4\bar{\alpha}_3\alpha_2(\bar{\alpha}_0\alpha_0 + \bar{\alpha}_1\alpha_1) - \alpha_{10}\bar{\alpha}_7\alpha_7\bar{\alpha}_7\alpha_6\bar{\alpha}_5\alpha_4\bar{\alpha}_3\alpha_2 - \underbrace{\alpha_{10}\bar{\alpha}_{10}}_{=0}\alpha_{10}\bar{\alpha}_7\alpha_6\bar{\alpha}_5\alpha_4\bar{\alpha}_3\alpha_2 \\
&= \alpha_{10}\bar{\alpha}_7\alpha_6\bar{\alpha}_5\alpha_4\bar{\alpha}_3\alpha_2(\bar{\alpha}_0\alpha_0 + \bar{\alpha}_1\alpha_1) + \alpha_{10}\bar{\alpha}_7\alpha_6\bar{\alpha}_5\alpha_4\bar{\alpha}_3\alpha_2\bar{\alpha}_2\alpha_2 \\
&= \alpha_{10}\bar{\alpha}_7\alpha_6\bar{\alpha}_5\alpha_4\bar{\alpha}_3\alpha_2(\underbrace{\bar{\alpha}_0\alpha_0 + \bar{\alpha}_1\alpha_1 + \bar{\alpha}_2\alpha_2}_{=0}) \\
&= 0.
\end{aligned} \tag{9.8}$$

We now describe the appearance of the knitting patterns obtained when  $n \geq 5$  and  $1 \leq i \leq n-2$  are arbitrary; the case  $n = 4$  is omitted but is entirely similar. When  $i = 1$  or  $i = 2$ , the knitting pattern always has the structure of a diagonal line of 1's from bottom-right to top-left, and the pattern terminates as in the  $i = 1$  and  $i = 2$  patterns from the  $n = 10$  case. In particular, the corresponding short exact sequences have left hand term  $V_i$  ( $i = 1$  or  $i = 2$ ) and middle terms  $V_0 \oplus V_{n-1}$  or  $V_0 \oplus V_1 \oplus V_{n-1}$ , respectively. If instead  $i \geq 3$  then the diagonal line of 1's stops with a circled 1 in the  $i^{\text{th}}$  row, followed by a  $-1$  one row down and one column to the left, as seen in the pattern corresponding to  $n = 10$ ,  $i = 3$ . Therefore the left hand term of the corresponding short exact sequence is  $V_i$  and the middle term is  $V_{i-1} \oplus V_{n-1}$ .

Finally, the third short exact sequence of (b) is obtained from one of the following patterns, where on the left  $n$  is even and on the right  $n$  is odd:

$$\begin{array}{ccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & \textcircled{1} & 0 & 0 \\
-1 & \boxed{1} & -1 & \boxed{1}
\end{array}$$

Possible choices for the maps in the short exact sequence are, when  $n$  is even,

$$\phi = \bar{\alpha}_n \quad \psi = \alpha_n,$$

and it is clear that their composition is 0.

Lastly, we consider the family of short exact sequences in (c). For the first and second of these, we again provide the knitting patterns in the cases  $n = 10$ ,  $i = 1, 2, 3$ , before describing the general pattern. They are, respectively:

$$\begin{array}{ccc}
\begin{array}{cccccccc}
\textcircled{0} & \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & 0 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \textcircled{1} & \boxed{1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{0} & \textcircled{1}
\end{array} &
\begin{array}{cccccccc}
\textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \textcircled{1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \textcircled{1} & \boxed{1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{0} & \textcircled{1}
\end{array} &
\begin{array}{cccccccc}
\textcircled{0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & \textcircled{1} & \boxed{1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{0} & \textcircled{1}
\end{array}
\end{array}$$

The claimed short exact sequences

$$\begin{aligned}
0 &\rightarrow V_1 \rightarrow V_0 \oplus V_9 \oplus V_{10} \rightarrow V_8 \rightarrow 0 \\
0 &\rightarrow V_2 \rightarrow V_0 \oplus V_1 \oplus V_9 \oplus V_{10} \rightarrow V_8 \rightarrow 0 \\
0 &\rightarrow V_3 \rightarrow V_2 \oplus V_9 \oplus V_{10} \rightarrow V_8 \rightarrow 0
\end{aligned} \tag{9.9}$$

then follow, and Lemma 8.5 applies to show that the maps between vertex modules are given by the shortest path between those vertices. We provide possible choices for the maps and the calculation that their composition is 0 for this final claim only in the case  $i = 2$ . Indeed, for this

case if we set

$$\phi = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_9 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \\ \alpha_{10} \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \end{pmatrix} \quad \psi = (\bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_0 \quad \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_1 \quad \bar{\alpha}_9 \quad \bar{\alpha}_{10})$$

then one calculates

$$\begin{aligned} \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_0 \cdot \alpha_0 + \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_1 \cdot \alpha_1 &= \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 (\bar{\alpha}_0 \alpha_0 + \bar{\alpha}_1 \alpha_1) \\ &= -\bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_2 \alpha_2 \\ &= \bar{\alpha}_7 \alpha_7 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \\ &= -(\bar{\alpha}_9 \alpha_9 + \bar{\alpha}_{10} \alpha_{10}) \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \\ &= -\bar{\alpha}_9 \cdot \alpha_9 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 - \bar{\alpha}_{10} \cdot \alpha_{10} \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \end{aligned} \tag{9.10}$$

and so  $\psi\phi = 0$ . For the general case, we use a similar analysis as for part (b): the only difference is the behaviour of the pattern at the start. This establishes the existence of the first and second short exact sequences in (c).

To establish the third and fourth short exact sequences in (c), the knitting patterns with  $n$  and  $j$  arbitrary and  $i = 1, 2, 3$ , where  $(i, j) \neq (1, 1)$ , have the following respective appearances:

$$\begin{array}{ccc} \begin{array}{ccccccc} \textcircled{0} & \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & 1 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \boxed{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{1} & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} & \begin{array}{ccccccc} \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \textcircled{1} & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \boxed{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{1} & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} & \begin{array}{ccccccc} \textcircled{0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \boxed{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{1} & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \end{array}$$

where here the boxed 1 appears in the  $j^{\text{th}}$  row of the pattern. We are able to read off the claimed short exact sequences, and use Lemma 8.5 to see that the maps between vertex modules correspond to the shortest path between those vertices. Calculating that  $\psi\phi = 0$  is similar to before. The fifth exact sequence of (c), which corresponds to the special case when  $(i, j) = (1, 1)$ , follows from the knitting pattern below:

$$\begin{array}{cc} \textcircled{0} & \textcircled{0} \\ -1 & \textcircled{1} & 1 \\ 0 & 0 \\ \vdots & \vdots \end{array}$$

(2) When proving the existence of these short exact sequences, we shall always assume that  $m = n$ , with the cases where  $m = n - 1$  being obtained symmetrically.

We establish the existence of the first short exact sequences in (a) and (b) using the illustrative examples of  $n = 10$  and  $n = 11$ , which result in the following completed knitting patterns:

$$\begin{array}{ccc} \begin{array}{cccccccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{1} & \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \boxed{1} & \cdot & \cdot \end{array} & \begin{array}{cccccccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{1} & \textcircled{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot \\ \cdot & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot \\ \cdot & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdot \\ -1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \boxed{1} & \cdot \end{array} \end{array}$$

For general  $n$ , the pattern is very similar: namely, we get a triangular pattern of 1's of height  $n - 1$ , and then the parity of  $n$  determines how the second bottom row and diagonal row of 1's on the left hand side of the pattern interact. This determines the placement of the  $-1$  entry and thus whether the left hand term in the short exact sequence is  $V_n$  or  $V_{n-1}$ . From the above patterns we are able to read off the claimed short exact sequences,

$$0 \rightarrow V_{10} \rightarrow V_0^{\oplus 2} \rightarrow V_{10} \rightarrow 0 \quad (9.11)$$

$$0 \rightarrow V_{11} \rightarrow V_0^{\oplus 2} \rightarrow V_{11} \rightarrow 0$$

and it remains to determine  $\phi$  and  $\psi$ , where we make use of Lemma 8.5. When  $n = 10$ , if we set

$$\phi = \begin{pmatrix} \alpha_0 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 \alpha_7 \bar{\alpha}_{10} \\ \alpha_0 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 \alpha_7 \bar{\alpha}_{10} \end{pmatrix} \quad \psi = (\alpha_{10} \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_0 \quad -\alpha_{10} \bar{\alpha}_9 \alpha_9 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_0) \quad (9.12)$$

then direct calculation shows that (where here we use without mention the identities  $\alpha_9 \bar{\alpha}_9 = 0$  and  $\alpha_{10} \bar{\alpha}_{10} = 0$ ),

$$\begin{aligned} & \alpha_{10} \bar{\alpha}_9 \alpha_9 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 \alpha_7 \bar{\alpha}_{10} \\ &= -\alpha_{10} \bar{\alpha}_9 \alpha_9 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 \alpha_7 \bar{\alpha}_{10} - \alpha_{10} \bar{\alpha}_9 \alpha_9 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 \alpha_7 \bar{\alpha}_{10} \\ &= \alpha_{10} \bar{\alpha}_7 \alpha_7 \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 \alpha_7 \bar{\alpha}_{10} + \alpha_{10} \bar{\alpha}_9 \alpha_9 (\bar{\alpha}_7 \alpha_7)^7 \bar{\alpha}_{10} \\ &= -\alpha_{10} \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 \alpha_7 \bar{\alpha}_{10} - \alpha_{10} \bar{\alpha}_9 \alpha_9 \bar{\alpha}_{10} \alpha_{10} (\bar{\alpha}_7 \alpha_7)^6 \bar{\alpha}_{10} \\ &= \alpha_{10} \bar{\alpha}_7 \alpha_6 \bar{\alpha}_5 \alpha_4 \bar{\alpha}_3 \alpha_2 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_2 \alpha_3 \bar{\alpha}_4 \alpha_5 \bar{\alpha}_6 \alpha_7 \bar{\alpha}_{10} \\ &\quad - \alpha_{10} \bar{\alpha}_9 \alpha_9 \bar{\alpha}_{10} \alpha_{10} \bar{\alpha}_9 \alpha_9 \bar{\alpha}_{10} \alpha_{10} \bar{\alpha}_9 \alpha_9 \bar{\alpha}_{10} \alpha_{10} \underbrace{\bar{\alpha}_9 \alpha_9 (\bar{\alpha}_9 \alpha_9 + \bar{\alpha}_{10} \alpha_{10}) \bar{\alpha}_{10}}_{=0} \end{aligned} \quad (9.13)$$

and so  $\psi\phi = 0$ . One obtains similar maps for arbitrary  $n$ .

If instead  $i = 1$ , then the pattern is similar to the  $i = 0$  case, except the middle of the pattern looks like

$$\begin{array}{ccccccc} & & \textcircled{0} & & \textcircled{1} & & \textcircled{0} \\ & 0 & \textcircled{0} & 1 & \textcircled{0} & 0 & \textcircled{0} \\ \cdots & & 1 & & 1 & & 0 & \cdots \\ & 1 & & 1 & & 1 & & \\ & & 1 & & 1 & & 1 & \\ \vdots & & & & & & & \ddots \end{array}$$

and so instead of having middle term  $V_0^{\oplus 2}$  we have middle term  $V_0 \oplus V_1$ . Moreover, the height of the triangle of 1's is now one less than the  $i = 0$  case, and this discrepancy means that the bottom of the pattern looks like that of the  $n = 11, i = 0$  pattern when  $n$  is even, and like that of the  $n = 10, i = 0$  pattern when  $n$  is odd; this explains how the parity of  $n - i$  affects the left hand terms in the corresponding short exact sequences. In this case, for each of the relevant vertices there are unique paths between them in the pattern, so we can read off  $\phi$  and  $\psi$ , up to sign.

Finally, for  $2 \leq i \leq n - 4$ , the only difference is that the middle of the pattern instead looks like

$$\begin{array}{ccccccc} & & & & \vdots & & \\ & & 0 & 0 & 0 & & \\ & & \textcircled{0} & \textcircled{1} & \textcircled{0} & & \\ \cdots & & 0 & 1 & 0 & & \cdots \\ & & 1 & & 1 & 0 & \\ & 1 & & 1 & & 1 & \\ \vdots & & & & & & \ddots \end{array}$$

where the circled 1 appears in the  $i^{\text{th}}$  row of the pattern, and a similar analysis explains why we obtain the claimed short exact sequences.  $\square$

Now fix some quasi-dominant weight  $\lambda$ . Since morphisms between vertex modules are given by left multiplication by paths between the corresponding vertices, we may view the short exact sequence of Proposition 9.1 as sequence of  $\mathcal{O}^\lambda$ -modules for any weight  $\lambda$ . The next step is to prove that a subset of these short exact sequences remain exact when working over  $\mathcal{O}^\lambda$  rather than  $R = \mathcal{O}$ .

**Corollary 9.14.** *Let  $\lambda$  be a quasi-dominant weight for  $\tilde{Q} = \tilde{\mathbb{D}}_n$ , and let  $Q'$  be a subquiver which is a connected component of  $Q_\lambda$ . Let  $\partial Q'$  be the set of all vertices of  $\tilde{Q}$  which are adjacent to  $Q'$ . Let  $\mathcal{S}$  be a short exact sequence of  $R$ -modules from Proposition 9.1 whose flanking terms correspond to vertices lying in  $Q'$  and whose middle terms correspond to vertices lying in  $\partial Q'$ . Then  $\mathcal{S}$  remains exact when viewed as a sequence of  $\mathcal{O}^\lambda$ -modules.*

*Proof.* By Lemma 8.7, it suffices to show that such a sequence is a complex, from which exactness will follow. That is, we must show that, given such a sequence, the composition  $\psi\phi$  is zero when viewed as  $\mathcal{O}^\lambda$ -module morphisms, instead of  $R$ -module morphisms. Ultimately, the reason that this holds is because, when showing  $\psi\phi = 0$  in  $\text{mod-}R$ , one only uses the relations at the vertices which survive when passing to  $Q_\lambda$ , and the weights at these vertices are equal to 0 in  $\mathcal{O}^\lambda$ ; that is, we have the same relations at these vertices as we do in  $R$ .

For example, consider the case when  $Q_\lambda$  has a connected component of type  $\mathbb{A}_3$ , with vertex set  $\{n-2, n-1, n\}$ , and so these vertices have weights  $\lambda_i = 0$ , and vertex  $n-3$  has a nonzero weight. We claim that (9.3), when  $n = 4$ , and (9.4), when  $n \geq 5$ , remain exact in these cases. For such a  $\lambda$ , when  $n = 4$  we have the following relevant relations in  $\Pi^\lambda$ ,

$$\alpha_3\bar{\alpha}_3 = 0, \quad \alpha_4\bar{\alpha}_4 = 0, \quad \bar{\alpha}_0\alpha_0 + \bar{\alpha}_1\alpha_1 + \bar{\alpha}_3\alpha_3 + \bar{\alpha}_4\alpha_4 = 0,$$

while when  $n \geq 5$  is even the relevant relations are instead

$$\alpha_{n-1}\bar{\alpha}_{n-1} = 0, \quad \alpha_n\bar{\alpha}_n = 0, \quad \bar{\alpha}_{n-3}\alpha_{n-3} + \bar{\alpha}_{n-1}\alpha_{n-1} + \bar{\alpha}_n\alpha_n.$$

The calculations in (9.5) and (9.6) only make use of these relations, so (9.3) and (9.4) are complexes, and hence exact by Lemma 8.7.

As another example, suppose instead that  $Q_\lambda$  has a type  $\mathbb{A}_{n-i}$  subquiver with vertices  $i, i+1, \dots, n-2, n$ ; for concreteness, and so that we can appeal to calculation (9.8), we take  $n = 10$  and  $i = 2$ . We now claim that (9.7) remains exact in  $\mathcal{O}^\lambda$ . Indeed, necessarily we must have zero weights at each vertex of  $Q_\lambda$ , and one verifies that calculation (9.8) only uses the relations at these vertices. Lemma 8.7 then proves the claim.

The argument proceeds in the same way for the other subquivers. For example, short exact sequences (9.9) and (9.11) correspond to certain type  $\mathbb{A}$  and type  $\mathbb{D}$  subquivers of  $Q_\lambda$ , respectively; one can check that calculations (9.10) and (9.13), respectively, only make use of relations at vertices which are present in these subquivers.  $\square$

In particular, this proves Lemma 5.4 in the Type  $\mathbb{D}$  case. Additionally, this allows us to prove Theorem 4.3 for this case:

*Proof of Theorem 4.3 in type  $\mathbb{D}$ .* Let  $\lambda$  be a quasi-dominant weight for  $\tilde{Q} = \tilde{\mathbb{D}}_n$  and suppose that  $Q'$  is a maximal connected component of  $Q_\lambda$ , and so necessarily  $Q'$  is either of type  $\mathbb{A}$  or type  $\mathbb{D}$ . We consider these two possibilities in turn.

First suppose that  $Q'$  is of type  $\mathbb{A}_m$ , and suppose that  $i$  and  $j$  are the vertices of  $Q'$  having valency 1 (unless  $m = 1$ , in which case  $i = j$  is the unique vertex of  $Q'$ ). Then, following the correspondence outlined in Remark 9.2, there is a unique short exact sequence in Proposition 9.1 (1) having flanking terms  $V_i$  and  $V_j$ . By Corollary 9.14, this sequence remains exact when viewed as a sequence in  $\text{mod-}\mathcal{O}^\lambda$ . Moreover, the middle terms correspond to vertices lying in  $\partial Q'$ , and are therefore either vertices with non-zero weights or the extending vertex; in either case, the middle term is a direct sum of projective modules, and hence projective. Therefore, passing to  $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda$ , we find that  $\Sigma V_i = V_j$  by definition. Since  $\Sigma$  induces a graph automorphism, it follows that  $\Sigma$  acts on the vertex modules corresponding to vertices in  $Q'$  as the Nakayama automorphism.

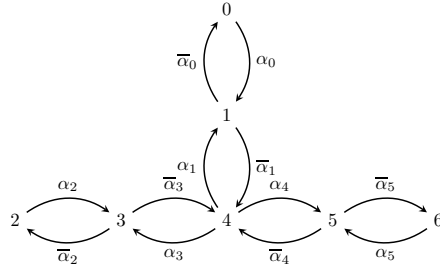
Now suppose the  $Q' = \mathbb{D}_4$ , so necessarily it has vertices  $n-3, n-2, n-1, n$ . Then precisely one of the short exact sequence of Proposition 9.1 (2a) has as its middle term a direct sum of vertex

modules corresponding to vertices in  $\partial Q'$ . This middle term is projective, so setting  $m = n - 1$  shows that  $\Sigma V_{n-1} = V_{n-1}$ , while setting  $m = n$  yields  $\Sigma V_n = V_n$ . Since  $\Sigma$  induces a graph automorphism, it permutes the vertex modules in  $Q'$  as claimed. If instead  $Q' = \mathbb{D}_{n-i}$ , with vertices  $i+1, i+2, \dots, n-2, n-1, n$  and where  $n-i \geq 5$ , then a similar argument as above shows that  $\Sigma$  behaves as claimed, where we appeal to part (2b) of Proposition 9.1 rather than part (2a) if  $n-i$  is odd. We also remark that we need only consider the corresponding short exact sequence when  $m = n$ , since an element of  $\text{Aut}(\mathbb{D}_k)$ , where  $k \geq 5$ , is uniquely determined by where it sends one of the branches of a  $\mathbb{D}_k$  graph.  $\square$

## 10. THE TRANSLATION FUNCTOR IN TYPE $\mathbb{E}$

We now turn our attention to determining the translation functor in the exceptional cases  $\mathbb{E}_6, \mathbb{E}_7$ , and  $\mathbb{E}_8$ . We will follow the same sequence of steps as in Section 9, but will omit some of the (entirely similar) details. We remark that, in the proof of Proposition 9.1, we provided knitting patterns for specific parameter choices, and then used these examples to explain the general appearance of the patterns for arbitrary  $n$ . On the other hand, since there are only finitely many subquivers of type  $\mathbb{E}$  quivers, the analogues of Proposition 9.1 consist of finitely many short exact sequences, and so we could explicitly prove the existence of all of them. For brevity, we do not; instead, we focus our attention on cases where determining the maps in the short exact sequences is more involved, and leave the remainder to the reader.

**10.1. Type  $\mathbb{E}_6$ .** Let  $\tilde{Q}$  be an extended Dynkin quiver of type  $\tilde{\mathbb{E}}_6$ , let  $Q = \mathbb{E}_6$  be the quiver obtained by deleting the extending vertex, and write  $R = \mathcal{O}(\tilde{Q})$ . We label the vertices and arrows of its double as in Figure 1, which we repeat below:



As before, we identify all possible connected subquivers which arise by deleting vertices from  $Q$ :

- (Type  $\mathbb{A}$ )  $Q = \mathbb{E}_6$  clearly has six  $\mathbb{A}_1$  subquivers, five  $\mathbb{A}_2$  subquivers, five  $\mathbb{A}_3$  subquivers, four  $\mathbb{A}_4$  subquivers, and one  $\mathbb{A}_5$  subquiver.
- (Type  $\mathbb{D}$ ) One can obtain both  $\mathbb{D}_4$  and  $\mathbb{D}_5$  subquivers: there is one  $\mathbb{D}_4$  subquiver, obtained by deleting vertices 2 and 6, and two  $\mathbb{D}_5$  subquivers, obtained by deleting vertex 2 or by deleting vertex 6.
- (Type  $\mathbb{E}$ ) There is clearly one type  $\mathbb{E}$  subquiver, namely  $Q = \mathbb{E}_6$  itself.

The following proposition is the analogue of Proposition 9.1 for type  $\mathbb{E}_6$ . It is split into three parts, each corresponding to a different Dynkin type, and these parts are then further subdivided by Roman numerals which indicate the number of vertices in the subquiver of interest. We have a similar correspondence between short exact sequences and subquivers as in the one outlined in Remark 9.2. In particular, the two short exact sequences of part (2iv) correspond to the (unique)  $\mathbb{D}_4$  subquiver of  $Q$ , but otherwise this correspondence is a bijection.

**Proposition 10.1.** *In the following, we work over  $R = \mathcal{O}(\tilde{Q})$ , where  $\tilde{Q} = \tilde{\mathbb{E}}_6$ .*

(1) (Type  $\mathbb{A}$  subquivers) There are short exact sequences

$$\begin{aligned}
 \text{(i)} \quad & 0 \rightarrow V_1 \rightarrow V_0 \oplus V_4 \rightarrow V_1 \rightarrow 0 \\
 & 0 \rightarrow V_2 \rightarrow V_3 \rightarrow V_2 \rightarrow 0 \\
 & 0 \rightarrow V_3 \rightarrow V_2 \oplus V_4 \rightarrow V_3 \rightarrow 0 \\
 & 0 \rightarrow V_4 \rightarrow V_1 \oplus V_3 \oplus V_5 \rightarrow V_4 \rightarrow 0 \\
 & 0 \rightarrow V_5 \rightarrow V_4 \oplus V_6 \rightarrow V_5 \rightarrow 0 \\
 & 0 \rightarrow V_6 \rightarrow V_5 \rightarrow V_6 \rightarrow 0 \\
 \text{(ii)} \quad & 0 \rightarrow V_1 \rightarrow V_0 \oplus V_3 \oplus V_5 \rightarrow V_4 \rightarrow 0 \\
 & 0 \rightarrow V_2 \rightarrow V_4 \rightarrow V_3 \rightarrow 0 \\
 & 0 \rightarrow V_3 \rightarrow V_1 \oplus V_2 \oplus V_5 \rightarrow V_4 \rightarrow 0 \\
 & 0 \rightarrow V_4 \rightarrow V_1 \oplus V_3 \oplus V_6 \rightarrow V_5 \rightarrow 0 \\
 & 0 \rightarrow V_5 \rightarrow V_4 \rightarrow V_6 \rightarrow 0 \\
 \text{(iii)} \quad & 0 \rightarrow V_1 \rightarrow V_0 \oplus V_2 \oplus V_5 \rightarrow V_3 \rightarrow 0 \\
 & 0 \rightarrow V_1 \rightarrow V_0 \oplus V_3 \oplus V_6 \rightarrow V_5 \rightarrow 0 \\
 & 0 \rightarrow V_2 \rightarrow V_1 \oplus V_5 \rightarrow V_4 \rightarrow 0 \\
 & 0 \rightarrow V_3 \rightarrow V_1 \oplus V_2 \oplus V_6 \rightarrow V_5 \rightarrow 0 \\
 & 0 \rightarrow V_4 \rightarrow V_1 \oplus V_3 \rightarrow V_6 \rightarrow 0 \\
 \text{(iv)} \quad & 0 \rightarrow V_1 \rightarrow V_0 \oplus V_5 \rightarrow V_2 \rightarrow 0 \\
 & 0 \rightarrow V_1 \rightarrow V_0 \oplus V_3 \rightarrow V_6 \rightarrow 0 \\
 & 0 \rightarrow V_2 \rightarrow V_1 \oplus V_6 \rightarrow V_5 \rightarrow 0 \\
 & 0 \rightarrow V_3 \rightarrow V_1 \oplus V_2 \rightarrow V_6 \rightarrow 0 \\
 \text{(v)} \quad & 0 \rightarrow V_2 \rightarrow V_1 \rightarrow V_6 \rightarrow 0
 \end{aligned}$$

(2) (Type  $\mathbb{D}$  subquivers) There are short exact sequences

$$\begin{aligned}
 \text{(iv)} \quad & 0 \rightarrow V_3 \rightarrow V_0 \oplus V_2^{\oplus 2} \oplus V_6 \rightarrow V_3 \rightarrow 0 \\
 & 0 \rightarrow V_5 \rightarrow V_0 \oplus V_2 \oplus V_6^{\oplus 2} \rightarrow V_5 \rightarrow 0 \\
 \text{(v)} \quad & 0 \rightarrow V_1 \rightarrow V_0^{\oplus 2} \oplus V_2^{\oplus 2} \rightarrow V_3 \rightarrow 0 \\
 & 0 \rightarrow V_1 \rightarrow V_0^{\oplus 2} \oplus V_6^{\oplus 2} \rightarrow V_5 \rightarrow 0
 \end{aligned}$$

(3) (Type  $\mathbb{E}$  subquivers) There is a short exact sequence

$$\text{(vi)} \quad 0 \rightarrow V_2 \rightarrow V_0^{\oplus 2} \rightarrow V_6 \rightarrow 0.$$

In (1), a map between vertex modules  $V_k$  and  $V_\ell$  is given by left multiplication by the shortest path from vertex  $\ell$  to vertex  $k$  in the double of  $\tilde{Q}$ , possibly with a change of sign. The maps in (2) and (3) will be given in (10.4) and (10.8).

*Proof.* Throughout the proof, given a short exact sequence of  $R$ -modules, we shall write  $\phi$  and  $\psi$  for the corresponding injection and surjection, respectively.

(1) These can all be directly computed using the knitting algorithm. The patterns corresponding to  $\mathbb{A}_1$  subquivers are easy to compute by hand, and in these cases the composition  $\psi\phi$  is simply  $\sum_{t(\alpha)=i}^{\alpha \in Q_0} \alpha \bar{\alpha} - \sum_{h(\alpha)=i}^{\alpha \in Q_0} \bar{\alpha} \alpha$ , and is therefore equal to 0. Rather than providing the relevant knitting patterns for all of the remaining cases, we consider a few specific examples, and leave the remainder to the reader. For example, we have the following knitting patterns,

$$\begin{array}{cccc}
 \begin{array}{c} \textcircled{1} \quad \textcircled{0} \\ -1 \textcircled{0} \quad 1 \textcircled{0} \quad 0 \\ \textcircled{0} \quad 0 \quad \textcircled{1} \textcircled{1} \textcircled{0} \\ \textcircled{0} \quad \textcircled{1} \quad \textcircled{0} \\ 0 \quad 0 \end{array} &
 \begin{array}{c} 0 \quad 0 \\ \textcircled{0} \textcircled{1} \textcircled{1} \textcircled{0} \textcircled{0} \\ -1 \quad 0 \quad 1 \quad 1 \quad 0 \\ 0 \quad 0 \quad \textcircled{1} \\ \textcircled{0} \quad \textcircled{1} \end{array} &
 \begin{array}{c} 0 \quad 0 \quad 0 \\ \textcircled{0} \textcircled{1} \textcircled{1} \textcircled{0} \textcircled{0} \textcircled{0} \\ -1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad \textcircled{1} \end{array} &
 \begin{array}{c} -1 \quad 1 \quad 0 \quad 0 \\ \textcircled{0} \quad 0 \quad \textcircled{1} \quad \textcircled{0} \quad \textcircled{0} \quad \textcircled{0} \\ 0 \quad \textcircled{0} \quad \textcircled{0} \quad \textcircled{1} \quad \textcircled{0} \quad \textcircled{0} \\ 0 \quad 0 \quad 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad \textcircled{1} \end{array}
 \end{array}$$

which, respectively, allow us to read off the short exact sequences

$$\begin{aligned}
 0 &\rightarrow V_3 \rightarrow V_1 \oplus V_2 \oplus V_5 \rightarrow V_4 \rightarrow 0 \\
 0 &\rightarrow V_1 \rightarrow V_0 \oplus V_3 \oplus V_6 \rightarrow V_5 \rightarrow 0 \\
 0 &\rightarrow V_1 \rightarrow V_0 \oplus V_3 \rightarrow V_6 \rightarrow 0 \\
 0 &\rightarrow V_2 \rightarrow V_1 \rightarrow V_6 \rightarrow 0.
 \end{aligned}$$

Using Lemma 8.5 we find that possible choices for the maps are given by



$$\begin{aligned}
\phi &= \begin{pmatrix} \bar{\alpha}_1 \alpha_3 \\ -\alpha_2 \\ \bar{\alpha}_4 \alpha_3 \end{pmatrix} & \psi &= (\alpha_1 \quad \alpha_3 \bar{\alpha}_2 \quad \alpha_4) \\
\phi &= \begin{pmatrix} \alpha_0 \\ \bar{\alpha}_3 \alpha_1 \\ \alpha_5 \bar{\alpha}_4 \alpha_1 \end{pmatrix} & \psi &= (\bar{\alpha}_4 \alpha_1 \bar{\alpha}_0 \quad \bar{\alpha}_4 \alpha_3 \quad \bar{\alpha}_5) \\
\phi &= \begin{pmatrix} \alpha_0 \\ -\bar{\alpha}_3 \alpha_1 \end{pmatrix} & \psi &= (\alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_0 \quad \alpha_5 \bar{\alpha}_4 \alpha_3) \\
\phi &= \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 & \psi &= \alpha_5 \bar{\alpha}_4 \alpha_1.
\end{aligned}$$

It is an easy calculation to verify that  $\psi\phi = 0$  in each of these cases.

(2) The following two knitting patterns,

$$\begin{array}{cccccc}
\textcircled{1} & \textcircled{0} & \textcircled{1} & & & \\
-1 \textcircled{0} & 1 \textcircled{1} & 0 \textcircled{0} & \textcircled{1} & & \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & \\
\textcircled{0} & \textcircled{1} & \textcircled{0} & & & 
\end{array}
\quad
\begin{array}{cccccc}
\textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{1} & & \\
0 \textcircled{1} & 0 \textcircled{0} & 1 \textcircled{1} & 0 \textcircled{0} & \textcircled{1} & \textcircled{1} \\
-1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & \\
0 & 0 & 1 & 0 & & & \\
0 & 0 & 1 & 0 & & & 
\end{array}$$

show that we have short exact sequences

$$0 \rightarrow V_3 \rightarrow V_0 \oplus V_2^{\oplus 2} \oplus V_6 \rightarrow V_3 \rightarrow 0 \quad (10.2)$$

$$0 \rightarrow V_1 \rightarrow V_0^{\oplus 2} \oplus V_2^{\oplus 2} \rightarrow V_3 \rightarrow 0, \quad (10.3)$$

and it remains to determine  $\phi$  and  $\psi$  using Lemma 8.5; indeed, if we set

$$\begin{aligned}
\phi &= \begin{pmatrix} \alpha_0 \bar{\alpha}_1 \alpha_3 \\ \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_3 \\ \alpha_2 \\ \alpha_5 \bar{\alpha}_4 \alpha_3 \end{pmatrix} & \psi &= (\bar{\alpha}_3 \alpha_1 \bar{\alpha}_0 \quad -\bar{\alpha}_2 \quad \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \quad -\bar{\alpha}_3 \alpha_4 \bar{\alpha}_5) \\
\phi &= \begin{pmatrix} \alpha_0 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_3 \alpha_1 \\ \alpha_0 \\ \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_1 \\ \alpha_2 \bar{\alpha}_3 \alpha_1 \end{pmatrix} & \psi &= (\bar{\alpha}_3 \alpha_1 \bar{\alpha}_0 \quad -\bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_0 \quad -\bar{\alpha}_2 \quad \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2)
\end{aligned} \quad (10.4)$$

then we calculate

$$\begin{aligned}
\bar{\alpha}_3 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_3 + \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \cdot \alpha_2 &= -\bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_1 \alpha_3 - \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_3 \alpha_3 \\
&= \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_3 \\
&= -\bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_3 - \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_3 \\
&= \bar{\alpha}_2 \cdot \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_3 + \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \cdot \alpha_5 \bar{\alpha}_4 \alpha_3,
\end{aligned} \quad (10.5)$$

and

$$\begin{aligned}
& \bar{\alpha}_3 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_3 \alpha_1 + \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \cdot \alpha_2 \bar{\alpha}_3 \alpha_1 \\
&= -\bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_3 \alpha_1 - \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_1 \\
&= \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_1 \\
&= -\bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_1 - \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_1 \\
&= \bar{\alpha}_2 \cdot \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_1 + \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_1 + \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_1 \\
&= \bar{\alpha}_2 \cdot \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_1 + \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 + \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \underbrace{\alpha_5 \bar{\alpha}_5}_{=0} \alpha_5 \bar{\alpha}_4 \alpha_1,
\end{aligned} \tag{10.6}$$

and so  $\psi\phi = 0$  in both cases. The remaining two short exact sequences are obtained symmetrically.

(3) Finally, we have the following completed knitting pattern,

$$\begin{array}{cccccccc}
-1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\oplus & 0 & \oplus & 1 & \oplus & 0 & \oplus & 1 & \oplus & 0 & \oplus & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}$$

from which we read off the short exact sequence

$$0 \rightarrow V_2 \rightarrow V_0^{\oplus 2} \rightarrow V_6 \rightarrow 0. \tag{10.7}$$

Again, we use Lemma 8.5 to determine the maps. Setting

$$\phi = \begin{pmatrix} \alpha_0 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \\ \alpha_0 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \end{pmatrix} \quad \psi = (\alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_0 \quad \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_1 \bar{\alpha}_0) \tag{10.8}$$

one calculates the following,

$$\begin{aligned}
& \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \\
&= -\alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_1 \alpha_3 \bar{\alpha}_2 \\
&= \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 + \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \\
&= \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \underbrace{\alpha_2 \bar{\alpha}_2}_{=0} \alpha_2 \bar{\alpha}_3 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 - \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 - \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \\
&= \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_1 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 + \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 + \underbrace{\alpha_5 \bar{\alpha}_5}_{=0} \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 \\
&= -\alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 - \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_2 - \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \\
&= -\alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_3 \bar{\alpha}_2 - \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \underbrace{\alpha_2 \bar{\alpha}_2}_{=0} - \alpha_5 \bar{\alpha}_4 \alpha_1 \bar{\alpha}_1 \alpha_4 \bar{\alpha}_5 \underbrace{\alpha_5 \bar{\alpha}_5}_{=0} \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2,
\end{aligned} \tag{10.9}$$

and so Lemma 8.5 tells us that we have found suitable choices for  $\phi$  and  $\psi$ .  $\square$

We now have the analogue of Corollary 9.14 for type  $\mathbb{E}_6$ :

**Corollary 10.10.** *Let  $\lambda$  be a quasi-dominant weight for  $\tilde{Q} = \tilde{\mathbb{E}}_6$ , and let  $Q'$  be a subquiver which is a connected component of  $Q_\lambda$ . Let  $\mathcal{S}$  be a short exact sequence of  $R$ -modules from Proposition 10.1 which corresponds to  $Q'$  as in Remark 9.2. Then  $\mathcal{S}$  remains exact when viewed as a sequence of  $\mathcal{O}^\lambda$ -modules.*

*Proof.* The general outline of the argument is the same as in the first paragraph of the proof of Corollary 9.14. We provide some of the details, focusing on the cases considered in the proof of Proposition 10.1, and leave the remainder to the reader.

Suppose that  $Q_\lambda$  has a connected component of type  $\mathbb{D}_4$ ; this forces  $Q_\lambda = \mathbb{D}_4$  with vertices

1, 3, 4, 5. This corresponds to having zero weights at vertices 1, 3, 4, 5, and so the following relations hold in  $\Pi^\lambda$ :

$$\bar{\alpha}_0\alpha_0 + \bar{\alpha}_1\alpha_1 = 0, \quad \bar{\alpha}_2\alpha_2 + \bar{\alpha}_3\alpha_3 = 0, \quad \alpha_1\bar{\alpha}_1 + \alpha_3\bar{\alpha}_3 + \alpha_4\bar{\alpha}_4 = 0, \quad \bar{\alpha}_4\alpha_4 + \bar{\alpha}_5\alpha_5 = 0.$$

These are the only relations that were used in calculation (10.5), and hence for such a choice of  $\lambda$  the short exact sequence of  $R$ -modules (10.2) is a complex when viewed as a sequence of  $\mathcal{O}^\lambda$ -modules, and hence is exact by Lemma 8.7.

Similarly, if  $Q_\lambda$  has as a connected component (in fact, is equal to) the  $\mathbb{D}_5$  subquiver with vertices 1, 3, 4, 5, 6, then necessarily the weights at these vertices are all equal to zero, and hence the relations at these vertices are the same as the relations at these vertices in  $\Pi(\tilde{Q})$ . But again, these are the only relations which are used in calculation (10.6), and so (10.3) remains exact when viewed as a sequence of  $\mathcal{O}^\lambda$ -modules by Lemma 8.7.

Lastly, the subquiver corresponding to (10.7) is all of  $Q_\lambda$  and has vertices 1, 2, 3, 4, 5, 6, and again this occurs precisely when the weights at these vertices are all zero. By the same line of reasoning as above, calculation (10.9) also holds in  $\Pi^\lambda(\tilde{Q})$ , and so (10.7) is exact as a sequence of  $\mathcal{O}^\lambda$ -modules.  $\square$

In particular, this proves Lemma 5.4 in the Type  $\mathbb{E}_6$  case. Additionally, this allows us to prove Theorem 4.3 for this case:

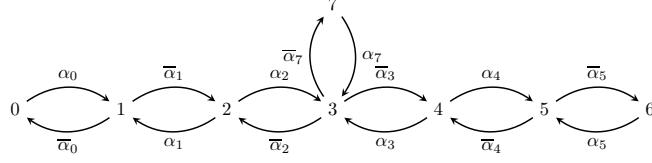
*Proof of Theorem 4.3 in type  $\mathbb{E}_6$ .* Let  $\lambda$  be a quasi-dominant weight for  $\tilde{Q} = \tilde{\mathbb{E}}_6$  and suppose that  $Q'$  is a connected component of  $Q_\lambda$ , and so necessarily  $Q'$  is of type  $\mathbb{A}_m$  ( $1 \leq m \leq 5$ ),  $\mathbb{D}_m$  ( $m = 4$  or  $5$ ), or  $\mathbb{E}_6$ . We consider each of these possibilities in turn.

First suppose that  $Q'$  is of type  $\mathbb{A}_m$ , and suppose that  $i$  and  $j$  are the vertices of  $Q'$  which have valency 1 (unless  $n = 1$ , in which case  $i = j$  is the unique vertex of  $Q'$ ). Then there is a unique short exact sequence in Proposition 10.1 (1) with flanking terms  $V_i$  and  $V_j$ . By Corollary 10.10, this short exact sequence remains exact in  $\mathcal{O}^\lambda$ . Moreover, the middle terms correspond to vertices lying in  $\partial Q'$ , and are therefore either vertices with non-zero weights or the 0 vertex; in either case, the middle term is a direct sum of projective modules. Therefore we find that  $\Sigma V_i = V_j$ , and since  $\Sigma$  induces a graph automorphism, it follows that  $\Sigma$  acts on the vertex modules of  $Q'$  as the Nakayama automorphism.

Now suppose that  $Q' = \mathbb{D}_4$ , so necessarily  $Q'$  has vertices 1, 3, 4, 5. Then the two short exact sequences of Proposition 10.1 (2iv) satisfy the hypotheses of Corollary 10.10 and so are exact when viewed as sequences of  $\mathcal{O}^\lambda$ -modules. Moreover, the middle terms are both projective, and so we deduce that  $\Sigma V_3 = V_3$  and  $\Sigma V_5 = V_5$ . Again, since  $\Sigma$  induces a graph automorphism of  $Q'$ , it follows that  $\Sigma$  acts as the identity on the vertex modules of  $Q' = \mathbb{D}_4$ ; that is, it acts as the Nakayama automorphism. If instead  $Q' = \mathbb{D}_5$ , then  $Q'$  has vertices 1, 2, 3, 4, 5 or 1, 3, 4, 5, 6; the two cases are symmetrical, so we consider only the second. In this case, the first short exact sequence in Proposition 10.1 (2v) remains exact over  $\mathcal{O}^\lambda$  and has a middle term which is projective, so  $\Sigma V_1 = V_3$ . Again, this completely determines how  $\Sigma$  acts on the vertex modules of  $Q' = \mathbb{D}_5$ ; namely, as the Nakayama automorphism.

Finally, suppose that  $Q' = \mathbb{E}_6$ . As above, the short exact sequence of Proposition 10.1 (3vi) remains exact over  $\mathcal{O}^\lambda$ , and has a middle term which is projective, so  $\Sigma V_2 = V_6$ . By Proposition 7.3, we again see that  $\Sigma$  acts as the Nakayama automorphism on the vertex modules of  $Q' = \mathbb{E}_6$ .  $\square$

**10.2. Type  $\mathbb{E}_7$ .** Let  $\tilde{Q}$  be an extended Dynkin quiver of type  $\tilde{\mathbb{E}}_7$ , let  $Q = \mathbb{E}_7$  be the quiver obtained by deleting the extending vertex, and write  $R = \mathcal{O}(\tilde{Q})$ . We label the vertices and arrows of its double as in Figure 1, which we repeat below:



By deleting vertices of  $Q$ , we can obtain the following subquivers:

- (Type  $\mathbb{A}$ )  $Q = \mathbb{E}_6$  clearly has seven  $\mathbb{A}_1$  subquivers, six  $\mathbb{A}_2$  subquivers, six  $\mathbb{A}_3$  subquivers, five  $\mathbb{A}_4$  subquivers, three  $\mathbb{A}_5$  subquivers, and one  $\mathbb{A}_6$  subquiver.
- (Type  $\mathbb{D}$ ) One can obtain  $\mathbb{D}_4$ ,  $\mathbb{D}_5$ , and  $\mathbb{D}_6$  subquivers. There is one  $\mathbb{D}_4$  subquiver, with vertices 2, 3, 4, 7, two  $\mathbb{D}_5$  subquivers, with vertices 1, 2, 3, 4, 7 and 2, 3, 4, 5, 7, and one  $\mathbb{D}_6$  subquiver, with vertices 2, 3, 4, 5, 6, 7.
- (Type  $\mathbb{E}$ ) There is a subquiver of type  $\mathbb{E}_6$ , obtained by deleting vertex 6, and a type  $\mathbb{E}_7$  subquiver, namely  $Q$  itself.

We now establish the analogue of Proposition 10.1, where we have a similar correspondence between subquivers of  $Q$  and short exact sequences. As before, we require two short exact sequences for the unique  $\mathbb{D}_4$  subquiver. We also note that there is no short exact sequence corresponding to the  $\mathbb{E}_7$  subquiver; while we could construct such an exact sequence, it would be superfluous for our needs.

**Proposition 10.11.** *In the following, we work over  $R = \mathcal{O}(\tilde{Q})$ , where  $\tilde{Q} = \tilde{E}_7$ .*

(1) (Type  $\mathbb{A}$  subquivers) *There are short exact sequences*

- |      |   |       |  |
|------|---|-------|--|
| (i)  | $0 \rightarrow V_1 \rightarrow V_0 \oplus V_2 \rightarrow V_1 \rightarrow 0$<br>$0 \rightarrow V_2 \rightarrow V_1 \oplus V_3 \rightarrow V_2 \rightarrow 0$<br>$0 \rightarrow V_3 \rightarrow V_2 \oplus V_4 \oplus V_7 \rightarrow V_3 \rightarrow 0$<br>$0 \rightarrow V_4 \rightarrow V_3 \oplus V_5 \rightarrow V_4 \rightarrow 0$<br>$0 \rightarrow V_5 \rightarrow V_4 \oplus V_6 \rightarrow V_5 \rightarrow 0$<br>$0 \rightarrow V_6 \rightarrow V_5 \rightarrow V_6 \rightarrow 0$<br>$0 \rightarrow V_7 \rightarrow V_3 \rightarrow V_7 \rightarrow 0$ | (iii) | $0 \rightarrow V_1 \rightarrow V_0 \oplus V_4 \oplus V_7 \rightarrow V_3 \rightarrow 0$<br>$0 \rightarrow V_2 \rightarrow V_1 \oplus V_5 \oplus V_7 \rightarrow V_4 \rightarrow 0$<br>$0 \rightarrow V_3 \rightarrow V_2 \oplus V_6 \oplus V_7 \rightarrow V_5 \rightarrow 0$<br>$0 \rightarrow V_4 \rightarrow V_3 \rightarrow V_6 \rightarrow 0$<br>$0 \rightarrow V_2 \rightarrow V_1 \oplus V_4 \rightarrow V_7 \rightarrow 0$<br>$0 \rightarrow V_4 \rightarrow V_2 \oplus V_5 \rightarrow V_7 \rightarrow 0$ |
| (ii) | $0 \rightarrow V_1 \rightarrow V_0 \oplus V_3 \rightarrow V_2 \rightarrow 0$<br>$0 \rightarrow V_2 \rightarrow V_1 \oplus V_4 \oplus V_7 \rightarrow V_3 \rightarrow 0$<br>$0 \rightarrow V_3 \rightarrow V_2 \oplus V_5 \oplus V_7 \rightarrow V_4 \rightarrow 0$<br>$0 \rightarrow V_4 \rightarrow V_3 \oplus V_6 \rightarrow V_5 \rightarrow 0$<br>$0 \rightarrow V_5 \rightarrow V_4 \rightarrow V_6 \rightarrow 0$<br>$0 \rightarrow V_3 \rightarrow V_2 \oplus V_4 \rightarrow V_7 \rightarrow 0$   | (iv)  | $0 \rightarrow V_1 \rightarrow V_0 \oplus V_5 \oplus V_7 \rightarrow V_4 \rightarrow 0$<br>$0 \rightarrow V_2 \rightarrow V_1 \oplus V_6 \oplus V_7 \rightarrow V_5 \rightarrow 0$<br>$0 \rightarrow V_3 \rightarrow V_2 \oplus V_7 \rightarrow V_6 \rightarrow 0$<br>$0 \rightarrow V_1 \rightarrow V_0 \oplus V_4 \rightarrow V_7 \rightarrow 0$<br>$0 \rightarrow V_5 \rightarrow V_2 \oplus V_6 \rightarrow V_7 \rightarrow 0$   |
|      |   | (v)   | $0 \rightarrow V_1 \rightarrow V_0 \oplus V_6 \oplus V_7 \rightarrow V_5 \rightarrow 0$<br>$0 \rightarrow V_2 \rightarrow V_1 \oplus V_7 \rightarrow V_6 \rightarrow 0$<br>$0 \rightarrow V_6 \rightarrow V_2 \rightarrow V_7 \rightarrow 0$   |
|      |   | (vi)  | $0 \rightarrow V_1 \rightarrow V_0 \oplus V_7 \rightarrow V_6 \rightarrow 0$   |

(2) (Type  $\mathbb{D}$  subquivers) *There are short exact sequences*

- |      |  |      |  |
|------|--|------|--|
| (iv) | $0 \rightarrow V_2 \rightarrow V_1^{\oplus 2} \oplus V_5 \rightarrow V_2 \rightarrow 0$<br>$0 \rightarrow V_4 \rightarrow V_1 \oplus V_5^{\oplus 2} \rightarrow V_4 \rightarrow 0$ | (v)  | $0 \rightarrow V_4 \rightarrow V_0 \oplus V_5^{\oplus 2} \rightarrow V_7 \rightarrow 0$<br>$0 \rightarrow V_2 \rightarrow V_1^{\oplus 2} \oplus V_6 \rightarrow V_7 \rightarrow 0$ |
|      |  | (vi) | $0 \rightarrow V_7 \rightarrow V_1^{\oplus 2} \rightarrow V_7 \rightarrow 0$   |

(3) (Type  $\mathbb{E}$  subquivers) *There is a short exact sequence*

$$(vi) \quad 0 \rightarrow V_1 \rightarrow V_0^{\oplus 2} \oplus V_6^{\oplus 2} \rightarrow V_5 \rightarrow 0$$

In (1), a map between vertex modules  $V_k$  and  $V_\ell$  is given by left multiplication by the shortest path from vertex  $\ell$  to vertex  $k$  in the double of  $\tilde{Q}$ , possibly with a change of sign. The maps in (2) and (3) will be given in (10.12) and (10.16).

*Proof.* Again, throughout the proof, given a short exact sequence of  $R$ -modules, we shall write  $\phi$  and  $\psi$  for the corresponding injection and surjection, respectively.

(1) The  $\mathbb{A}_1$  cases are established as in the proof of Proposition 10.1. For the remaining cases, we have, for example, the following completed knitting patterns:

$$\begin{array}{ccccc}
 \begin{array}{ccc} \textcircled{0} & \textcircled{0} & \\ 0 & 0 & 0 \\ \textcircled{1} & \textcircled{0} & \\ -1 & \textcircled{1} & \textcircled{0} \\ 0 & \textcircled{1} & \textcircled{0} \\ \textcircled{0} & \textcircled{1} & \textcircled{0} \\ 0 & 0 & \end{array} &
 \begin{array}{ccc} \textcircled{0} & \textcircled{0} & \\ 0 & 0 & 0 \\ \textcircled{1} & \textcircled{0} & \\ -1 & \textcircled{1} & \textcircled{0} \\ 0 & 0 & 1 \\ 0 & 0 & \textcircled{1} \\ 0 & \textcircled{1} & \end{array} &
 \begin{array}{ccc} \textcircled{1} & \textcircled{0} & \textcircled{0} \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & \textcircled{0} & \textcircled{1} \\ 0 & 0 & \textcircled{1} \\ \textcircled{0} & \textcircled{0} & \textcircled{1} \\ 0 & 0 & 0 \end{array} &
 \begin{array}{ccc} \textcircled{1} & \textcircled{0} & \textcircled{0} \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & \textcircled{0} & \textcircled{1} \\ 0 & 0 & 1 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & \textcircled{1} \end{array} &
 \begin{array}{ccc} \textcircled{1} & \textcircled{0} & \textcircled{0} \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & \textcircled{0} & \textcircled{1} \\ 0 & 0 & 1 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & \textcircled{1} \end{array}
 \end{array}$$

and we read off the following short exact sequences,

$$\begin{aligned}
 0 &\rightarrow V_3 \rightarrow V_2 \oplus V_5 \oplus V_7 \rightarrow V_4 \rightarrow 0 \\
 0 &\rightarrow V_3 \rightarrow V_2 \oplus V_6 \oplus V_7 \rightarrow V_5 \rightarrow 0 \\
 0 &\rightarrow V_1 \rightarrow V_0 \oplus V_5 \oplus V_7 \rightarrow V_4 \rightarrow 0 \\
 0 &\rightarrow V_1 \rightarrow V_0 \oplus V_6 \oplus V_7 \rightarrow V_5 \rightarrow 0 \\
 0 &\rightarrow V_1 \rightarrow V_0 \oplus V_7 \rightarrow V_6 \rightarrow 0.
 \end{aligned}$$

The knitting algorithm also determines the maps, up to sign, and possible choices for these are, respectively,

$$\begin{aligned}
 \phi &= \begin{pmatrix} \alpha_2 \\ \overline{\alpha}_4 \alpha_3 \\ \alpha_7 \end{pmatrix} & \psi &= (\alpha_3 \overline{\alpha}_2 \quad -\alpha_4 \quad \alpha_3 \overline{\alpha}_7) \\
 \phi &= \begin{pmatrix} \alpha_2 \\ \alpha_5 \overline{\alpha}_4 \alpha_3 \\ \alpha_7 \end{pmatrix} & \psi &= (\overline{\alpha}_4 \alpha_3 \overline{\alpha}_2 \quad \overline{\alpha}_5 \quad \overline{\alpha}_4 \alpha_3 \overline{\alpha}_7) \\
 \phi &= \begin{pmatrix} \alpha_0 \\ \overline{\alpha}_4 \alpha_3 \overline{\alpha}_2 \alpha_1 \\ \alpha_7 \overline{\alpha}_2 \alpha_1 \end{pmatrix} & \psi &= (\alpha_3 \overline{\alpha}_2 \alpha_1 \overline{\alpha}_0 \quad -\alpha_4 \quad -\alpha_3 \overline{\alpha}_7) \\
 \phi &= \begin{pmatrix} \alpha_0 \\ \alpha_5 \overline{\alpha}_4 \alpha_3 \overline{\alpha}_2 \alpha_1 \\ \alpha_7 \overline{\alpha}_2 \alpha_1 \end{pmatrix} & \psi &= (\overline{\alpha}_4 \alpha_3 \overline{\alpha}_2 \alpha_1 \overline{\alpha}_0 \quad \overline{\alpha}_5 \quad \overline{\alpha}_4 \alpha_3 \overline{\alpha}_7) \\
 \phi &= \begin{pmatrix} \alpha_0 \\ \alpha_7 \overline{\alpha}_2 \alpha_1 \end{pmatrix} & \psi &= (\alpha_5 \overline{\alpha}_4 \alpha_3 \overline{\alpha}_2 \alpha_1 \overline{\alpha}_0 \quad \alpha_5 \overline{\alpha}_4 \alpha_3 \overline{\alpha}_7).
 \end{aligned}$$

The easy calculations showing that  $\psi\phi = 0$  are omitted.

(2) The knitting algorithm generates the following knitting patterns,

$$\begin{array}{ccc}
 \begin{array}{cccc} \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} \\ \textcircled{1} & \textcircled{0} & \textcircled{1} & \textcircled{0} \\ -1 & 1 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} &
 \begin{array}{cccc} \textcircled{0} & \textcircled{0} & \textcircled{1} & \textcircled{0} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} &
 \begin{array}{cccc} \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} \\ \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{1} \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}
 \end{array}$$

and so we have short exact sequences

$$\begin{aligned}
0 \rightarrow V_2 \rightarrow V_1^{\oplus 2} \oplus V_5 \rightarrow V_2 \rightarrow 0 \\
0 \rightarrow V_4 \rightarrow V_0 \oplus V_5^{\oplus 2} \rightarrow V_7 \rightarrow 0 \\
0 \rightarrow V_7 \rightarrow V_1^{\oplus 2} \rightarrow V_7 \rightarrow 0
\end{aligned}$$

where the maps are, respectively, given by

$$\begin{aligned}
\phi &= \begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \\ \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \end{pmatrix} & \psi &= \begin{pmatrix} -\alpha_2 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_1 & \alpha_1 & \alpha_2 \bar{\alpha}_3 \alpha_4 \end{pmatrix} \\
\phi &= \begin{pmatrix} \alpha_0 \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \\ \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \\ \bar{\alpha}_4 \end{pmatrix} & \psi &= \begin{pmatrix} \alpha_7 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 & -\alpha_7 \bar{\alpha}_3 \alpha_4 & \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_3 \alpha_4 \end{pmatrix} \\
\phi &= \begin{pmatrix} \bar{\alpha}_1 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_7 \\ \bar{\alpha}_1 \alpha_2 \bar{\alpha}_7 \end{pmatrix} & \psi &= \begin{pmatrix} \alpha_7 \bar{\alpha}_2 \alpha_1 & \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \end{pmatrix}.
\end{aligned} \tag{10.12}$$

The relevant calculations which show that  $\psi\phi = 0$  in each case are as follows:

$$\begin{aligned}
& -\alpha_2 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_1 \cdot \bar{\alpha}_1 + \alpha_1 \cdot \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 + \alpha_2 \bar{\alpha}_3 \alpha_4 \cdot \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \\
& \quad = \alpha_2 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 - \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 - \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \\
& \quad = \alpha_2 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 + \alpha_2 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \\
& \quad = \alpha_2 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \\
& \quad = 0,
\end{aligned} \tag{10.13}$$

$$\begin{aligned}
& \alpha_7 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 - \alpha_7 \bar{\alpha}_3 \alpha_4 \cdot \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 + \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_3 \alpha_4 \cdot \bar{\alpha}_4 \\
& \quad = -\alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 + \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 - \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \\
& \quad = -\alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 + \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 + \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \\
& \quad = -\alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 - \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \\
& \quad = \alpha_7 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \\
& \quad = 0,
\end{aligned} \tag{10.14}$$

$$\begin{aligned}
& \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \cdot \bar{\alpha}_1 \alpha_2 \bar{\alpha}_7 = -\alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_7 \\
& \quad = \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \\
& \quad = -\alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 - \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \\
& \quad = \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 + \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \\
& \quad = -\alpha_7 \bar{\alpha}_2 \alpha_1 \cdot \bar{\alpha}_1 \alpha_2 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 + \alpha_7 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \underbrace{\alpha_5 \bar{\alpha}_5}_{=0} \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_7.
\end{aligned} \tag{10.15}$$

The remaining two short exact sequences are obtained symmetrically.

(3) Finally, the last short exact sequence

$$0 \rightarrow V_1 \rightarrow V_0^{\oplus 2} \oplus V_6^{\oplus 2} \rightarrow V_5 \rightarrow 0$$

comes from the knitting pattern

$$\begin{array}{ccccccc}
\textcircled{1} & \textcircled{0} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & \\
-1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & \square \\
\textcircled{0} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & \textcircled{1} & 
\end{array}$$

and possible choices for the maps are

$$\phi = \begin{pmatrix} \alpha_0 \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \\ \alpha_0 \\ \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \\ \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_1 \end{pmatrix} \quad (10.16)$$

$$\psi = \begin{pmatrix} -\bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 & \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 & -\bar{\alpha}_5 & \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \end{pmatrix}.$$

Indeed, one calculates that

$$\begin{aligned}
& \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 + \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_4 \bar{\alpha}_5 \cdot \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_1 \\
&= \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_1 - \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \\
&= -\bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 - \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \\
&= \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \\
&= -\bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 - \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \\
&= \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 + \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 \\
&= \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1 + \bar{\alpha}_5 \cdot \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_7 \alpha_7 \bar{\alpha}_2 \alpha_2 \bar{\alpha}_3 \alpha_3 \bar{\alpha}_2 \alpha_1,
\end{aligned} \quad (10.17)$$

and so  $\psi\phi = 0$ .  $\square$

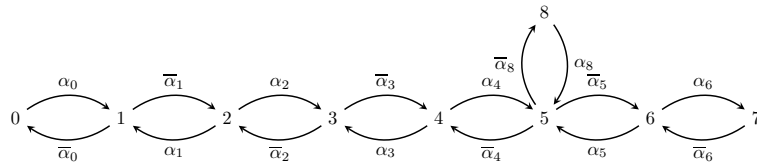
**Corollary 10.18.** *Let  $\lambda$  be a quasi-dominant weight for  $\tilde{Q} = \tilde{\mathbb{E}}_7$ , and let  $Q'$  be a subquiver which is a connected component of  $Q_\lambda$ . Let  $\mathcal{S}$  be a short exact sequence of  $R$ -modules from Proposition 10.11 which corresponds to  $Q'$  as in Remark 9.2. Then  $\mathcal{S}$  remains exact when viewed as a sequence of  $\mathcal{O}^\lambda$ -modules.*

*Proof.* This is essentially the same argument as the proof of Corollary 10.10. The reader is encouraged to verify that calculations (10.13)-(10.17) only use relations at vertices  $i$  with  $\lambda_i = 0$ .  $\square$

As with Corollary 10.10, this establishes Lemma 5.4 in the Type  $\mathbb{E}_7$  case.

*Proof of Theorem 4.3 in type  $\mathbb{E}_7$ .* Again one simply adapts the arguments of the proof of Theorem 4.3 for the type  $\mathbb{E}_6$  case. We remark that, in the case of a weight  $\lambda$  giving rise to an  $\mathbb{E}_7$  subquiver, since  $\Sigma$  induces a graph automorphism and  $\text{Aut}(\mathbb{E}_7)$  is trivial, we immediately deduce that  $\Sigma$  acts as the Nakayama automorphism in this case.  $\square$

**10.3. Type  $\mathbb{E}_8$ .** Let  $\tilde{Q}$  be an extended Dynkin quiver of type  $\tilde{\mathbb{E}}_8$ , let  $Q = \mathbb{E}_8$  be the quiver obtained by deleting the extending vertex, and write  $R = \mathcal{O}(\tilde{Q})$ . We label the vertices and arrows of its double as in Figure 1, which we repeat below:



The possible subquivers of  $Q$  are as follows:

- (Type  $\mathbb{A}$ )  $Q = \mathbb{E}_6$  clearly has eight  $\mathbb{A}_1$  subquivers, seven  $\mathbb{A}_2$  subquivers, seven  $\mathbb{A}_3$  subquivers, six  $\mathbb{A}_4$  subquivers, four  $\mathbb{A}_5$  subquivers, three  $\mathbb{A}_6$  subquivers, and one  $\mathbb{A}_7$  subquiver.
- (Type  $\mathbb{D}$ ) One can obtain  $\mathbb{D}_4$ ,  $\mathbb{D}_5$ ,  $\mathbb{D}_6$ , and  $\mathbb{D}_7$  subquivers. There is one  $\mathbb{D}_4$  subquiver, with vertices 4, 5, 6, 8, two  $\mathbb{D}_5$  subquivers, with vertices 3, 4, 5, 6, 8 and 4, 5, 6, 7, 8, one  $\mathbb{D}_6$  subquiver, with vertices 2, 3, 4, 5, 6, 8, and one  $\mathbb{D}_7$  subquiver, with vertices 1, 2, 3, 4, 5, 6, 8.
- (Type  $\mathbb{E}$ ) There is a subquiver of type  $\mathbb{E}_6$ , obtained by deleting vertices 1, 2, one subquiver of type  $\mathbb{E}_7$ , obtained by deleting vertex 1, and a type  $\mathbb{E}_8$  subquiver, namely  $Q$  itself.

As before, each short exact sequence in the following proposition corresponds to a Dynkin subquiver; notably, there are two exact sequences for the  $\mathbb{D}_4$  subquiver, and none for the  $\mathbb{E}_7$  and  $\mathbb{E}_8$  subquivers.

**Proposition 10.19.** *In the following, we work over  $R = \mathcal{O}(\tilde{Q})$ , where  $\tilde{Q} = \tilde{E}_8$ .*

(1) (Type  $\mathbb{A}$  subquivers) *There are short exact sequences*

$$\begin{array}{ll}
\text{(i)} \quad 0 \rightarrow V_1 \rightarrow V_0 \oplus V_2 \rightarrow V_1 \rightarrow 0 & \text{(iii)} \quad 0 \rightarrow V_1 \rightarrow V_0 \oplus V_4 \rightarrow V_3 \rightarrow 0 \\
\quad 0 \rightarrow V_2 \rightarrow V_1 \oplus V_3 \rightarrow V_2 \rightarrow 0 & \quad 0 \rightarrow V_2 \rightarrow V_1 \oplus V_5 \rightarrow V_4 \rightarrow 0 \\
\quad 0 \rightarrow V_3 \rightarrow V_2 \oplus V_4 \rightarrow V_3 \rightarrow 0 & \quad 0 \rightarrow V_3 \rightarrow V_2 \oplus V_6 \oplus V_8 \rightarrow V_5 \rightarrow 0 \\
\quad 0 \rightarrow V_4 \rightarrow V_3 \oplus V_5 \rightarrow V_4 \rightarrow 0 & \quad 0 \rightarrow V_4 \rightarrow V_3 \oplus V_7 \oplus V_8 \rightarrow V_6 \rightarrow 0 \\
\quad 0 \rightarrow V_5 \rightarrow V_4 \oplus V_6 \oplus V_8 \rightarrow V_5 \rightarrow 0 & \quad 0 \rightarrow V_5 \rightarrow V_4 \oplus V_8 \rightarrow V_7 \rightarrow 0 \\
\quad 0 \rightarrow V_6 \rightarrow V_5 \oplus V_7 \rightarrow V_6 \rightarrow 0 & \quad 0 \rightarrow V_4 \rightarrow V_3 \oplus V_6 \rightarrow V_8 \rightarrow 0 \\
\quad 0 \rightarrow V_7 \rightarrow V_6 \rightarrow V_7 \rightarrow 0 & \quad 0 \rightarrow V_6 \rightarrow V_4 \oplus V_7 \rightarrow V_8 \rightarrow 0 \\
\quad 0 \rightarrow V_8 \rightarrow V_5 \rightarrow V_8 \rightarrow 0 & \text{(iv)} \quad 0 \rightarrow V_1 \rightarrow V_0 \oplus V_5 \rightarrow V_4 \rightarrow 0 \\
\text{(ii)} \quad 0 \rightarrow V_1 \rightarrow V_0 \oplus V_3 \rightarrow V_2 \rightarrow 0 & \quad 0 \rightarrow V_2 \rightarrow V_1 \oplus V_6 \oplus V_8 \rightarrow V_5 \rightarrow 0 \\
\quad 0 \rightarrow V_2 \rightarrow V_1 \oplus V_4 \rightarrow V_3 \rightarrow 0 & \quad 0 \rightarrow V_3 \rightarrow V_2 \oplus V_7 \oplus V_8 \rightarrow V_6 \rightarrow 0 \\
\quad 0 \rightarrow V_3 \rightarrow V_2 \oplus V_5 \rightarrow V_4 \rightarrow 0 & \quad 0 \rightarrow V_4 \rightarrow V_3 \oplus V_8 \rightarrow V_7 \rightarrow 0 \\
\quad 0 \rightarrow V_4 \rightarrow V_3 \oplus V_6 \oplus V_8 \rightarrow V_5 \rightarrow 0 & \quad 0 \rightarrow V_3 \rightarrow V_2 \oplus V_6 \rightarrow V_8 \rightarrow 0 \\
\quad 0 \rightarrow V_5 \rightarrow V_4 \oplus V_7 \oplus V_8 \rightarrow V_6 \rightarrow 0 & \quad 0 \rightarrow V_7 \rightarrow V_4 \rightarrow V_8 \rightarrow 0 \\
\quad 0 \rightarrow V_6 \rightarrow V_5 \rightarrow V_7 \rightarrow 0 & \text{(v)} \quad 0 \rightarrow V_1 \rightarrow V_0 \oplus V_6 \oplus V_8 \rightarrow V_5 \rightarrow 0 \\
\quad 0 \rightarrow V_5 \rightarrow V_4 \oplus V_6 \rightarrow V_8 \rightarrow 0 & \quad 0 \rightarrow V_2 \rightarrow V_1 \oplus V_7 \oplus V_8 \rightarrow V_6 \rightarrow 0 \\
& \quad 0 \rightarrow V_3 \rightarrow V_2 \oplus V_8 \rightarrow V_7 \rightarrow 0 \\
& \quad 0 \rightarrow V_2 \rightarrow V_1 \oplus V_6 \rightarrow V_8 \rightarrow 0 \\
& \text{(vi)} \quad 0 \rightarrow V_1 \rightarrow V_0 \oplus V_7 \oplus V_8 \rightarrow V_6 \rightarrow 0 \\
& \quad 0 \rightarrow V_2 \rightarrow V_1 \oplus V_8 \rightarrow V_7 \rightarrow 0 \\
& \quad 0 \rightarrow V_1 \rightarrow V_0 \oplus V_6 \rightarrow V_8 \rightarrow 0 \\
& \text{(vii)} \quad 0 \rightarrow V_1 \rightarrow V_0 \oplus V_8 \rightarrow V_7 \rightarrow 0
\end{array}$$

(2) (Type  $\mathbb{D}$  subquivers) *There are short exact sequences*

$$\begin{array}{ll}
\text{(iv)} \quad 0 \rightarrow V_4 \rightarrow V_3^{\oplus 2} \oplus V_7 \rightarrow V_4 \rightarrow 0 & \text{(vi)} \quad 0 \rightarrow V_8 \rightarrow V_1 \oplus V_7^{\oplus 2} \rightarrow V_8 \rightarrow 0 \\
\quad 0 \rightarrow V_6 \rightarrow V_3 \oplus V_7^{\oplus 2} \rightarrow V_6 \rightarrow 0 & \text{(vii)} \quad 0 \rightarrow V_6 \rightarrow V_0 \oplus V_7^{\oplus 3} \rightarrow V_8 \rightarrow 0 \\
\text{(v)} \quad 0 \rightarrow V_6 \rightarrow V_2 \oplus V_7^{\oplus 2} \rightarrow V_8 \rightarrow 0 & \\
\quad 0 \rightarrow V_4 \rightarrow V_3^{\oplus 2} \rightarrow V_8 \rightarrow 0 &
\end{array}$$

(3) (Type  $\mathbb{E}$  subquivers) *There is a short exact sequence*

$$\text{(vi)} \quad 0 \rightarrow V_3 \rightarrow V_2^{\oplus 2} \rightarrow V_7 \rightarrow 0$$

In (1), a map between vertex modules  $V_k$  and  $V_\ell$  is given by left multiplication by the shortest path from vertex  $\ell$  to vertex  $k$  in the double of  $\tilde{Q}$ , possibly with a change of sign. The maps in (2) and (3) will be given in (10.20) and (10.22).

*Proof.* Again, we shall write  $\phi$  and  $\psi$  for the injection and surjection in a short exact sequence, respectively.



(1) The arguments proving the existence of the short exact sequences corresponding to type  $\mathbb{A}$  subquivers are very similar to those in the preceding two subsections, so we leave them to the diligent reader. In any case, the resulting maps between vertex modules are given by the shortest path between those vertices, and a quick calculation determines the correct sign choices.

(2) We have the following knitting patterns:

$$\begin{array}{ccc}
 \begin{array}{ccccccc}
 \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & & & \\
 & 0 & 0 & 0 & 0 & 0 & \\
 0 & & 0 & 0 & 0 & & \\
 \textcircled{1} & & & \textcircled{1} & \textcircled{0} & & \\
 -1 & 1 & 0 & & \boxed{1} & & \\
 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & & 1 & 0 & & \\
 \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & & & 
 \end{array} &
 \begin{array}{ccccccc}
 \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \textcircled{0} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & & \\
 0 & 0 & 0 & 1 & 0 & 0 & \\
 0 & 0 & 1 & 1 & 0 & & \\
 0 & 0 & 0 & 1 & 1 & 0 & \boxed{1} & 0 \\
 -1 & 1 & 0 & & 1 & 0 & \\
 \textcircled{1} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & & 
 \end{array} &
 \begin{array}{ccccccc}
 \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \\
 \textcircled{1} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & & \\
 -1 & 1 & 0 & & 1 & 0 & \\
 0 & 0 & 0 & 1 & 1 & 0 & \boxed{1} & 0 \\
 0 & 0 & & 1 & 1 & 0 & \\
 0 & 0 & 0 & 1 & 0 & 0 & 
 \end{array} \\
 \\
 \begin{array}{ccccccc}
 \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \\
 \textcircled{0} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \\
 0 & 0 & 0 & 1 & 0 & 0 & \\
 0 & 0 & 1 & 1 & 0 & 0 & \\
 0 & 0 & 1 & 1 & 1 & 0 & \\
 -1 & 0 & 1 & 1 & 0 & 1 & \boxed{1} & 0 \\
 0 & 0 & 1 & 0 & 1 & 0 & \\
 \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & 
 \end{array} &
 \begin{array}{ccccccc}
 \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & \textcircled{0} \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
 0 & 0 & 0 & 1 & 1 & 0 & 1 & \boxed{1} & 0 \\
 -1 & 1 & 0 & 1 & 0 & 1 & 0 & \\
 \textcircled{1} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} 
 \end{array}
 \end{array}$$

and therefore we have short exact sequences

$$\begin{aligned}
 0 &\rightarrow V_4 \rightarrow V_3^{\oplus 2} \oplus V_7 \rightarrow V_4 \rightarrow 0 \\
 0 &\rightarrow V_6 \rightarrow V_2 \oplus V_7^{\oplus 2} \rightarrow V_8 \rightarrow 0 \\
 0 &\rightarrow V_4 \rightarrow V_3^{\oplus 2} \rightarrow V_8 \rightarrow 0 \\
 0 &\rightarrow V_8 \rightarrow V_1 \oplus V_7^{\oplus 2} \rightarrow V_8 \rightarrow 0 \\
 0 &\rightarrow V_6 \rightarrow V_0 \oplus V_7^{\oplus 3} \rightarrow V_8 \rightarrow 0
 \end{aligned}$$

(the proof of the existence of the second exact sequence in part (2) of the statement of the proposition is symmetrical to that of the first exact sequence). One checks that the correct maps are, respectively,

$$\begin{aligned}
 \phi &= \begin{pmatrix} \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \\ \bar{\alpha}_3 \\ \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \end{pmatrix} & \psi &= (\alpha_3 \quad -\alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_3 \quad \alpha_4 \bar{\alpha}_5 \alpha_6) \\
 \phi &= \begin{pmatrix} \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \\ \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \\ \bar{\alpha}_6 \end{pmatrix} & \psi &= (\alpha_8 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \quad \alpha_8 \quad -\alpha_8 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \alpha_6) \\
 \phi &= \begin{pmatrix} \bar{\alpha}_3 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \\ \bar{\alpha}_3 \end{pmatrix} & \psi &= (\alpha_8 \bar{\alpha}_4 \alpha_3 \quad \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_3) \\
 \phi &= \begin{pmatrix} \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_8 \\ \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_8 \\ \bar{\alpha}_4 \alpha_5 \bar{\alpha}_8 \end{pmatrix} & \psi &= (\alpha_8 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_1 \quad -\alpha_8 \bar{\alpha}_5 \alpha_6 \quad \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_4) \\
 \phi &= \begin{pmatrix} \alpha_0 \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 \\ \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \bar{\alpha}_5 \\ \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 \\ \bar{\alpha}_6 \end{pmatrix} & \psi &= \begin{pmatrix} \alpha_8 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 \\ \alpha_8 \bar{\alpha}_5 \alpha_6 \\ \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_6 \\ -\alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_6 \end{pmatrix}^T
 \end{aligned} \tag{10.20}$$

The calculations which verify that these are valid choices, in the sense that the relevant compositions are zero, are similar to those in the previous two subsections. We omit all but the last of these, which is the most involved:

$$\begin{aligned}
& \alpha_8 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \alpha_1 \bar{\alpha}_0 \cdot \alpha_0 \bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_5 + \alpha_8 \bar{\alpha}_5 \alpha_6 \cdot \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 \\
&= \alpha_8 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 - \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 \\
&= \alpha_8 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 - \alpha_8 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 \\
&= -\alpha_8 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 \tag{10.21} \\
&= -\alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 \\
&= \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 + \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 \\
&= -\alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_6 \cdot \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 - \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \\
&= -\alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_6 \cdot \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_5 + \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_6 \bar{\alpha}_6.
\end{aligned}$$

(3) Finally, the last short exact sequence follows from the following pattern:

$$\begin{array}{cccccccc}
\textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\textcircled{1} & \textcircled{0} & \textcircled{0} & \textcircled{1} & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} \\
-1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \square
\end{array}$$

and we claim that the corresponding maps are

$$\phi = \begin{pmatrix} \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_3 \\ \alpha_2 \end{pmatrix} \quad \psi = (\bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \quad \bar{\alpha}_6 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2). \tag{10.22}$$

By Lemma 8.5, it suffices to verify that  $\psi\phi = 0$ . Indeed,

$$\begin{aligned}
& \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \cdot \alpha_2 \bar{\alpha}_3 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_3 \\
&= -\bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_3 \\
&= \bar{\alpha}_6 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_3 + \bar{\alpha}_6 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_3 \\
&= \bar{\alpha}_6 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_3 - \underbrace{\bar{\alpha}_6 \alpha_6}_{=0} \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_4 \alpha_3 \tag{10.23} \\
&= -\bar{\alpha}_6 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_4 \bar{\alpha}_4 \alpha_3 - \bar{\alpha}_6 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_3 \\
&= -\bar{\alpha}_6 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_5 \alpha_5 \bar{\alpha}_4 \alpha_3 \bar{\alpha}_2 \cdot \alpha_2 - \bar{\alpha}_6 \alpha_5 \bar{\alpha}_8 \alpha_8 \bar{\alpha}_5 \alpha_6 \underbrace{\bar{\alpha}_6 \alpha_6}_{=0} \bar{\alpha}_6 \alpha_5 \bar{\alpha}_4 \alpha_3,
\end{aligned}$$

as required.  $\square$

**Corollary 10.24.** *Let  $\lambda$  be a quasi-dominant weight for  $\tilde{Q} = \tilde{\mathbb{E}}_8$ , and let  $Q'$  be a subquiver which is a connected component of  $Q_\lambda$ . Let  $\mathcal{S}$  be a short exact sequence of  $R$ -modules from Proposition 10.11 which corresponds to  $Q'$  as in Remark 9.2. Then  $\mathcal{S}$  remains exact when viewed as a sequence of  $\mathcal{O}^\lambda$ -modules.*

*Proof.* This is essentially the same argument as the proof of Corollary 10.10. For example, one can check that calculations (10.21) and (10.23) only use relations at vertices  $i$  with  $\lambda_i = 0$ .  $\square$

This also establishes Lemma 5.4 in the Type  $\mathbb{E}_8$  case.

*Proof of Theorem 4.3 in type  $\mathbb{E}_8$ .* Again, to prove this simply adapt the proof of Theorem 4.3 for the type  $\mathbb{E}_6$  case. The remark at the end of the proof of Theorem 4.3 for the type  $\mathbb{E}_7$  case also applies here for  $\mathbb{E}_7$  and  $\mathbb{E}_8$  subquivers.  $\square$

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